Random matrices and enumeration of maps

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Abstract. We review recent developments in random matrix theory related with the enumeration of connected oriented graphs called maps. In particular, we show that the long standing use of matrix integrals in physics to tackle such issues can be made rigorous and discuss some applications. This talk is based on joint works with E. Maurel-Segala and O. Zeitouni.

1. Introduction

A map is a connected oriented diagram which can be embedded into a surface. Its genus g is by definition the smallest genus of a surface in which it can be embedded in such a way that edges do not cross and the faces of the graph (which are defined by following the boundary of the graph) are homeomorphic to a disc. One has the formula for the Euler characteristic χ:

$$\chi = 2 - 2g = \sharp \text{ vertices} + \sharp \text{ faces} - \sharp \text{ edges}.$$
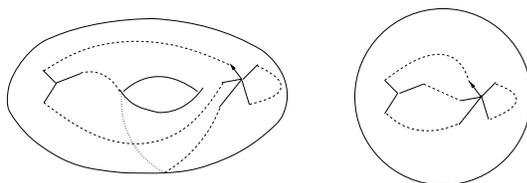

Figure 1. Examples of maps with 2 vertices of degree 3 and 5 respectively, with $g = 1$ and $g = 0$.

In the sequel, we shall be interested in the enumeration of maps up to equivalence classes, namely up to homeomorphisms of the oriented surface. This amounts to consider the purely combinatorial problem of enumerating the possible arrangements of the edges of graphs with prescribed vertices and genus. A dual point of view goes as follows. We can replace each vertex with valence k by a face whose boundary is made of k edges (each of them crossing a different edge adjacent to the vertex). The problem can then be reformulated as the enumeration of possible tilings of a surface

of given genus by a given number of faces with prescribed degree (the degree of the face is the number of edges that border the face). For some problems (for instance related with statistical mechanics), one would like eventually to color the edges or the vertices of the map and impose additional constraints for the gluing of edges/vertices of different colors.

The problem of enumerating maps was first tackled in the sixties by W. Tutte [34], [35] who was motivated by combinatorial problems such as the four color problem (see [26] or [4] for combinatorial motivations and problems). Tutte considered *rooted* planar maps. The root of a map is a distinguished oriented edge. Fixing a root allows to reduce the number of symmetries of the problem; enumerating rooted maps is equivalent to count maps with labelled edges. Tutte showed that diverse 'chirurgical' operations on rooted planar maps allow to obtain equations for the generating functions of the numbers of these maps with a given number of faces, each face having the same fixed degree. One of the examples of the maps which were exactly enumerated by Tutte are triangulations (i.e maps with faces of degree 3); he proved [34] that the number of rooted triangulations with $2n$ faces is given by $2^{n+1}(3n)!/(2n+2)!n!$ (see e.g. E. Bender and E. Canfield [5] for generalizations). In general, the equations obtained by Tutte's approach are not exactly solvable; their analysis was the subject of subsequent developments (see e.g. [19]).

Because this last problem is in general difficult, a bijective approach was developed after the work of R. Cori and B. Vauquelin [13] and G. Schaeffer's thesis (see e.g. [32]). It was shown that planar triangulations and quadrangulations can be encoded by labelled trees, which are much easier to count. This idea proved to be very fruitful in many respects. It allows not only to study the number of maps but also part of their geometry; P. Chassaing and G. Schaeffer [12] could prove that the diameter of uniformly distributed quadrangulations with n vertices behaves like $n^{\frac{1}{4}}$. This technique was first applied to triangulations or quadrangulations, but soon generalized to other maps, see e.g. [10] or [9]. The case of planar bi-colored maps related to the so-called Ising model on random planar graphs could also be studied [7]. Further, it allows also to tackle maps with higher genus, an avenue recently opened by M. Marcus and G. Schaeffer. In general, this approach give more complete results than the other methods. However, it yet can not cover all the models which were analysed in physics by the so-called matrix models approach and when it does, the solution for the enumeration problem has the same flavour than the solution obtained with matrix models (see [9]).

The question of enumerating maps has been studied intensively in physics for more than thirty years. One of the first motivation came in QCD (which stands for Quantum Chromodynamics) with a large number N of colors; 't Hooft [33] noticed in the seventies that as N is large, physical quantities can be expanded, via Feynamnn diagrams, as sums over maps. This fundamental remark allowed the connection between quantum field theory and the problem of enumerating maps, and in particular led to the use of matrix integrals to count maps (In [11], this technique was used to

enumerate planar maps with vertices of degree 4). The interest in enumerating maps was revived by quantum gravity in the eighties; random triangulations could be used for instance to approximate fluctuating geometries. As a side product, people got interested by statistical models defined on random graphs. Such models should in fact be related at criticality with the corresponding model on $\mathbb{Z}^2$ (see [28]). Maps were also used to approximate low-dimensional string theory (see e.g. the review [15]). Although recently the methods introduced by R. Cori, B. Vauquelin and G. Scheaffer began to be developed in physics too (by P. Di Francesco et al.), the most common approach has been to use matrix models, a rather indirect but quite powerful method that we shall describe in this survey (see also A. Zvonkin [36]). It is based on the particular form of Gaussian moments as given by Wick formula; if $(G_1, \ldots, G_{2n})$ is a centered Gaussian vector, then Wick formula asserts that

$$\mathbb{E}[G_1 G_2 \ldots G_{2n}] = \sum_{\substack{1 \leq s_1 < s_2 .. < s_n \leq 2n \\ r_i > s_i}} \prod_{j=1}^{n} \mathbb{E}[G_{s_j} G_{r_j}].$$

Alternatively, this formula can be represented by Feynman diagrams. Let us now consider matrices from the Gaussian Unitary Ensemble (GUE). For a fixed dimension N, let $\mathcal{H}_N$ be the set of $N \times N$ Hermitian matrices. The law of the GUE is then given as the Gaussian law on $\mathcal{H}_N$

$$\mu_N(dA) = \frac{1}{Z_N} e^{-\frac{N}{2} \operatorname{tr}(A^2)} dA.$$

In other words, $A_{lk} = \bar{A}_{kl}$ for $1 \leq k < l \leq N$ and

$$A_{kl} = (2N)^{-\frac{1}{2}} (g_{kl}^1 + i g_{kl}^2) \quad \text{for } k < l, \quad A_{kk} = N^{-\frac{1}{2}} g_{kk}^1,$$

where the $(g_{kl}^1, g_{kl}^2, k \leq l)$ are independent identically distributed standard Gaussian variables. One can then observe that Wick formula implies that for all integer numbers $p_i, 1 \leq i \leq k$, all $k \in \mathbb{N}$, $N \in \mathbb{N}$,

$$\int \prod_{i=1}^{k} (N \operatorname{tr}(A^{p_i})) \, d\mu_N(A) = \sum_{F \geq 0} N^{F+k-\frac{\sum p_i}{2}} G((p_i)_{1 \leq i \leq k}, F) \qquad (1)$$

with

$$G((p_i)_{1 \leq i \leq k}, F) = \sharp \{\text{oriented graphs with } F \text{ faces and}$$
$$1 \text{ vertex of degree } p_i, 1 \leq i \leq k \}.$$

In $G((p_i)_{1 \leq i \leq k}, F)$, the edges of the graph are labelled. One should notice that the number $F + k - \frac{\sum p_i}{2}$ corresponds to $2 - 2g$, with g the genus of the surface on which a connected oriented graph with F faces and one vertex of degree p_i for $1 \leq i \leq k$

can be embedded, since such a graph has k vertices and $2^{-1} \sum p_i$ edges. Hence, if we see the dimension N of the matrices as a parameter, the expectation of the trace of moments of matrices from the GUE can be seen as a generating function for the number of oriented graphs with a given genus and a given number of vertices with prescribed degree. Laplace transforms of traces of matrices from the GUE should therefore be generating functions for maps. In fact, we find, by expanding the exponential and using (1) that, with $t = (t_1, \ldots, t_k)$,

$$\log Z_N(t) := \log \int e^{-\sum_{i=1}^{k} t_i N \operatorname{tr}(A^{p_i})} d\mu_N(A) = \sum_{g \geq 0} N^{2-2g} F_g(t) \qquad (2)$$

with

$$F_g(t) := \sum_{n_1, \ldots, n_k \in \mathbb{N}^k} \prod_{i=1}^{k} \frac{(-t_i)^{k_i}}{k_i!} M((p_i, n_i)_{1 \leq i \leq k}; g)$$

the generating function for the number $M((p_i, n_i)_{1 \leq i \leq k}; g)$ of maps with genus g and n_i vertices of degree p_i for $i \in \{1, \ldots, k\}$. Note here that we now count maps, and so oriented graphs that are connected, due to the fact that we took the logarithm. Formula (2) is only formal, i.e means that all the derivatives of the functions on each side of the equality match at $(t_1, \ldots, t_k) = (0, \ldots, 0)$.

An interesting feature of the relation (1) is that it can be generalized to several matrices, corresponding then to the enumeration of colored-edges maps. Namely, let us introduce a bijection between non-commutative monomials and oriented vertices with colored half-edges and a distinguished half-edge as follows; to the letters $(X_1, \ldots, X_m)$ we associate half-edges with m different colors $c_1, \ldots, c_m$, and to a monomial $q(X_1, \ldots, X_m) = X_{i_1} \ldots X_{i_k}$ a clockwise oriented vertex with first half-edge (which is distinguished) of color c_{i_1}, second of color c_{i_2} till the last half-edge of color c_{i_k}. We call such a vertex, equipped with its colored half-edges, orientation and distinguished edge, a star of type q. It defines a bijection between monomials and stars. We then can generalize (2) as follows; let $(q_1, \ldots, q_k)$ be k non-commutative monomials of m indeterminates, then

$$\int \prod_{i=1}^{k} (N \operatorname{tr}(q_i(A_1, \ldots, A_m))) \, d\mu_N(A_1) \ldots d\mu_N(A_m)$$

$$= \sum_{F \geq 0} N^{k+F - \frac{\sum p_i}{2}} G_c((q_i)_{1 \leq i \leq k}, F), \qquad (3)$$

with $G_c((q_i)_{1 \leq i \leq k}, F)$ the number of oriented graphs with F faces and one star of type q_i, $1 \leq i \leq k$, the gluing between half-edges of different colors being forbidden. (2) also generalizes to this multi-matrix setting and we find that

$$\log Z_N(t) = \log \int e^{-N \sum_{i=1}^{k} t_i \operatorname{tr}(q_i(A_1, \ldots, A_m))} \prod_{i=1}^{m} d\mu_N(A_i) \qquad (4)$$

expends formally as a generating function of colored maps (here $M((p_i, n_i), 1 \leq i \leq k; g)$ has to be replaced by the number $M_c((q_i, n_i), 1 \leq i \leq k; g)$ of maps with genus g and n_i stars of type q_i, the gluing between half-edges of different colors being forbidden.)

As we said before, these considerations have been intensively used in physics to analyze various combinatorial models via their representations in terms of matrices. It is no surprise that mathematicians end up wondering what physicists are doing or come to cross the same lines of thoughts. In the last ten years, progress in the theory of random matrices led to a better mathematical understanding of this approach. The first natural question is to find a reasonable domain of the parameters $(t_1, \ldots, t_n)$ where the expansion (2) or (4) are not only formal. Because the right hand side is a priori a diverging series, this expansion can not be obtained analytically in a neighborhood of the origin, but we would like to show that equality holds up to some error term N^{-2k} provided the parameters belong to some neighborhood of the origin. Once this question is settled, one can try to 'solve'(and then in which sense?) the combinatorial problem by estimating the matrix integral.

As we shall see in the next section, the first goal has received a rather complete answer in the last few years. For the second question and one matrix setting, it turns out that at least the first order asymptotics of the left-hand-side of (2) can be computed by using standard saddle point (or large deviations) techniques. The answer is yet not very transparent since it is given by a variational formula and we shall review part of its analysis. In the multi-matrix setting, very few results have been obtained so far, a few of which we shall describe.

2. Expansion of the free energy of matrix models

2.1. One matrix integrals. In the case of one matrix, the free energy of the matrix model can be expressed as an integral over the eigenvalues of the random matrix. It is well known (see [30]) that the law of the eigenvalues of the GUE can be described by a Coulomb gas law;

$$d\sigma_N(\lambda_1, \ldots, \lambda_N) = \frac{1}{Z_N} \prod_{i<j} |\lambda_i - \lambda_j|^2 e^{-\frac{N}{2} \sum_{i=1}^{N} (\lambda_i)^2} \prod d\lambda_i$$

with Z_N the normalizing constant. It therefore turns out that (2) is given by

$$Z_N(t) = Z_N^{-1} \int e^{-N \sum_{i=1}^{N} [V_t(\lambda_i) + \frac{1}{2}(\lambda_i)^2]} \prod_{i<j} |\lambda_i - \lambda_j|^2 \prod d\lambda_i \qquad (5)$$

with $V_t(x) = \sum_{i=1}^{k} t_i x^{p_i}$. To make sure that $Z_N(t)$ is finite for each $N \in \mathbb{N}$, we shall assume that $p_k = \max_{l \leq k} p_l$ is even and $t_k > 0$. It was conjectured by Bessis, Itzykson and Zuber [6] that $\log Z_N(t)$ can be expanded in the vicinity of the origin. It

was only twenty years later that this question met its complete mathematical treatment in [17] (see also [2], [1] for previous advances in the subject). It was indeed shown in [17], Theorem 1.1, that if we let

$$\mathbb{T}(T, \gamma) = \left\{ t \in \mathbb{R}^k : \sum_{i=1}^{k} |t_i| \leq T, \ t_k > \gamma \sum_{i=1}^{k-1} |t_i| \right\},$$

then the following result holds:

Theorem 2.1. *For all $k \in \mathbb{N}$, there is $T > 0$ and $\gamma > 0$ so that for $t \in \mathbb{T}(T, \gamma)$, for all $k \in \mathbb{N}$, one has the expansion*

$$N^{-2} \log Z_N(t) = \sum_{l=0}^{k} N^{-2l} F_l(t) + O(N^{-2k-2}).$$

Moreover,

$$F_l(t) = \sum_{n_1,\ldots,n_k \in \mathbb{N}^n} \prod_{l=1}^{k} \frac{(-t_i)^{n_i}}{n_i!} M((p_i, n_i), 1 \leq i \leq k; g)$$

with $M((p_i, n_i)_{1 \leq i \leq k}; g)$ the number of maps with genus g and n_i vertices of degree p_i for $1 \leq i \leq k$.

This result is based on an expansion for the mean empirical density of the eigenvalues under the associated Gibbs measure

$$d\sigma_t^N(\lambda_1, \ldots, \lambda_N) = Z_N(t)^{-1} e^{-N \sum_{i=1}^{N} V_t(\lambda_i)} \prod_{i<j} |\lambda_i - \lambda_j|^2 e^{-\frac{N}{2} \sum_{i=1}^{N} (\lambda_i)^2} \prod d\lambda_i.$$

Indeed, if we set μ_t^N to be the probability measure on $\mathbb{R}$ given for any bounded measurable test function f by

$$\int f(x) d\mu_t^N(x) = \int \frac{1}{N} \sum_{i=1}^{N} f(\lambda_i) \, d\sigma_t^N(\lambda_1, \ldots, \lambda_N),$$

it is proved in [17], Theorem 1.3, that

Theorem 2.2. *For all $k \in \mathbb{N}$, there is $T > 0$ and $\gamma > 0$ so that for $t \in \mathbb{T}(T, \gamma)$, for all $k \in \mathbb{N}$, one has the expansion*

$$\int f(x) d\mu_t^N(x) = \sum_{g=0}^{k} N^{-2g} f_g(t) + O(N^{-2k-2})$$

for any smooth function f which grows no faster than a polynomial at infinity. Moreover, for all $p \in \mathbb{N}$, if $f(x) = x^p$,

$$f_g(t) = \sum_{n_1,\ldots,n_k \in \mathbb{N}^n} \prod_{l=1}^{k} \frac{(-t_i)^{n_i}}{n_i!} M((p, 1), (p_i, n_i)_{1 \leq i \leq k}; g).$$

Note that this second theorem implies the first since for all l,

$$\partial_{t_l} \log Z_N(\boldsymbol{t}) = \mu_{\boldsymbol{t}}^N(x^{p_l})$$

gives

$$\log \frac{Z_N(\boldsymbol{t})}{Z_N(\boldsymbol{0})} = \sum_{l=1}^{k} \int_0^{t_l} \mu_{(0,\ldots,0,s,t_{l+1},\ldots,t_k)}^N(x^{p_l})\, ds.$$

The proof of these results are based on orthogonal polynomials; because the interaction between the eigenvalues $(\lambda_1, \ldots, \lambda_N)$ are given in terms of the square of a Vandermonde determinant, the density of the law $\mu_{\boldsymbol{t}}^N$ can be expressed in terms of orthogonal polynomials, whose limits are well known (since they are completely integrable). The theory of integrable systems allowed many important breakthroughs in the theory of matrix models, but we want to argue in the next section that the large N expansion of matrix models can be obtained by more direct arguments.

2.2. Many matrix integrals. In [22], [23], [29], following (3), we considered the multi-matrix integral defined, for k non-commutative monomials of m indeterminates $(q_1, \ldots, q_k)$, by

$$Z_N(\boldsymbol{t}) = \int e^{-N \sum_{i=1}^{k} t_i \, \mathrm{tr}(q_i(A_1,\ldots,A_m))}\, d\mu_N(A_1)\ldots d\mu_N(A_m). \tag{6}$$

To make this integral finite and not oscillatory, we assume the following. Let $*$ be the involution on polynomial functions of m non-commutative integrals given by

$$(z X_{i_1} \ldots X_{i_p})^* = \bar{z} X_{i_p} \ldots X_{i_1}$$

for any $p \in \mathbb{N}$ and any $i_j \in \{1, \ldots, m\}$. Then, to avoid possible oscillations, we assume that $V_{\boldsymbol{t}}(X_1, \ldots, X_m) = \sum_{i=1}^{k} t_i q_i(X_1, \ldots, X_m)$ is self-adjoint, i.e. $V_{\boldsymbol{t}} = V_{\boldsymbol{t}}^*$. To bound the integral, we assume that there exists $c > 0$ so that $V_{\boldsymbol{t}}$ is c-convex, i.e $W(X_1, \ldots, X_m) = V_{\boldsymbol{t}}(X_1, \ldots, X_m) + \frac{(1-c)}{2} \sum_{i=1}^{k} X_i^2$ is convex in the sense that for any $N \in \mathbb{N}$, the application

$$(X_1, \ldots, X_m) \in \mathcal{H}_N^m \to \mathrm{tr}(W(X_1, \ldots, X_m))$$

is a convex function of the entries of $(X_1, \ldots, X_m)$. Observe that, by Klein's lemma, if V is a convex function of one real variable, V is convex in the above sense and therefore our condition includes all $V_{\boldsymbol{t}}$ of the form

$$V_{\boldsymbol{t}}(X_1, \ldots, X_m) = \sum V_i\Big(\sum \kappa_i^j X_j\Big) + \sum \beta_{jl} X_i X_j$$

with V_i convex functions of one variable, real numbers κ_i^j and β_{jl} small enough constant (depending on c). This assumption generalizes that of Theorem 2.2 since if γ is large enough and T small enough, for $\boldsymbol{t} \in \mathbb{T}(T, \gamma)$, the potential $V_{\boldsymbol{t}}(x) =$

$\sum_{i=1}^{k} t_i x^{p_i} + 2^{-1}(1 - c)x^2$ is strictly convex. In [22], [23], [29], the analogue of Theorems 2.1 and 2.2 were obtained for a range of parameters which are small enough and so that V_t stays uniformly c-convex for some $c > 0$. For the analogue of Theorem 2.2, μ_t^N is generalized into the linear form on non-commutative polynomials given by

$$\mu_t^N(P) = \frac{1}{Z_N(t)} \int \frac{1}{N} \operatorname{tr}(P(A_1, \dots, A_m)) e^{-N \sum_{l=1}^{k} t_i \operatorname{tr}(q_i(A_1,\dots,A_m))} \prod d\mu_N(A_i).$$

The techniques are complectly different from those of [17] (in fact, orthogonal polynomial techniques are unknown for general multi-matrix models) and rely on combinatorial interpretations of non-commutative differential operators. For instance, for the first order expansion, it can be shown under our hypothesis that for any non-commutative polynomial P, $\mu_t^N(P)$ converges towards some quantity $\tau(P)$. Furthermore, τ satisfies some 'non-commutative differential equation', called the Schwinger–Dyson equation, which says that for all polynomials P,

$$\tau((X_i + D_i V_t)P) = \tau \otimes \tau(\partial_i P), \quad \tau(1) = 1 \tag{7}$$

with ∂_i (resp. D_i) the non-commutative derivative (resp. the cyclic derivative) given on a monomial P by

$$\partial_i P = \sum_{P=P_1 X_i P_2} P_1 \otimes P_2, \quad D_i P = \sum_{P=P_1 X_i P_2} P_2 P_1$$

where the sums run over all the possible decomposition of the monomial P into $P_1 X_i P_2$. It turns out that in our range of parameters, there is only one solution to (7) (which satisfies some boundedness properties that the limit points of μ_t^N share), which corresponds to the generating function for planar maps. This last identification comes out because ∂_i and D_i have very simple combinatorial interpretations; if you think of τ as the generating function of maps, you will see that ∂_i consists in the operation of splitting your map into two disjoint maps when two edges of the color c_i of one vertex are glued together, whereas D_i will consist in the operation of erasing one edge when two different vertices are connected via an edge of the color c_i, then obtaining a single bigger vertex (see [22]). Amazingly, it turns out that these non-commutative derivatives play exactly the same role than the surgery initially introduced by Tutte. One could then wonder what matrix models brought so far. At least a funny remark; the limit τ, whose moments are generating functions for maps, is a tracial state. In particular, in the one matrix case, it is a probability measure. Another remark is that the higher orders in the expansion of matrix integrals can be expressed in terms of τ and the differential operators defined by the ∂_i and the D_i. Thus the expansion describes, without further thinking, the operations that one can do on a map of genus g to enumerate it in terms of lower genus maps. In the next section, we shall therefore concentrate on planar maps and the analysis of the limiting state τ.

3. Estimating matrix integrals

We shall focus in this section on the first order of matrix integrals, that is on planar maps.

3.1. One matrix integrals. It is easily seen by a saddle point method (or large deviations, see e.g. [3]) that with $Z_N(t)$ given by (5) and $V_t(x) = \sum_{i=1}^{k} t_i x^{p_i}$, we have

$$\lim_{N \to \infty} \frac{1}{N^2} \log Z_N(t) = \sup_{\mu \in \mathcal{P}(\mathbb{R})} \left\{ \int \log |x-y| \, d\mu(x) d\mu(y) - \int \left(V_t(x) + \frac{x^2}{2} \right) d\mu(x) \right\}$$
(8)

up to a universal constant coming from the limit of $N^{-2} \log Z_N$. Moreover, the above supremum is achieved at a unique probability measure μ_t and we have for all bounded continuous function,

$$\lim_{N \to \infty} \mu_t^N(f) = \mu_t(f).$$

In particular, at least for $t \in \mathbb{T}(T, \gamma)$ as in Theorem 2.2, for all integer p, $\mu_t(x^p)$ is a generating function for maps with one vertex of degree p and so at least formally,

$$G\mu_t(z) := \int \frac{1}{z-x} \, d\mu_t(x)$$

$$= \frac{1}{z} \sum_{l \geq 0} z^{-l} \sum_{n_1, \ldots, n_k} \prod_{i=1}^{k} \frac{(-t_i)^{n_i}}{n_i!} M((l, 1), (p_i, n_i)_{1 \leq i \leq k}; 0)$$

is a generating function for maps too. In [14] (see also [17]), the solution to the variational problem (8) has been studied. It turns out that in the small range of parameters we are considering, we have the following characterization of μ_t;

$$\mu_t(dx) = \frac{1_{[\alpha(t), \beta(t)]}}{2\pi} \sqrt{(x - \alpha(t))(\beta(t) - x)} h_t(x) dx$$

with h_t a polynomial given explicitly in terms of V_t and $\alpha(t)$, $\beta(t)$ determined by the set of equations

$$\int_{\alpha(t)}^{\beta(t)} \frac{(V_t'(s) + s)}{\sqrt{(s - \alpha(t))(\beta(t) - s)}} ds = 0, \quad \int_{\alpha(t)}^{\beta(t)} \frac{s(V_t'(s) + s)}{\sqrt{(s - \alpha(t))(\beta(t) - s)}} ds = 2\pi.$$

$\alpha(t)$, $\beta(t)$ are analytic functions of $t \in \mathbb{T}(T, \gamma)$. This however does not give a very explicit formula for $G\mu_t$. When V_t is a monomial, more detailed analysis were performed in [14]. It turns out that when V_t is even (see [9]), the analysis is more simple and, in the case $V_t = tx^4$, can be pushed to obtain explicit formulas (see [11]).

3.2. Many matrix integrals. The problem of enumerating colored, or decorated, maps is much more challenging. In combinatorics, only the so-called Ising model on random quadrangulations could be tackled so far (see [7]). The list of models which could be 'solved' in physics is slightly longer; it includes for instance the so called Potts model, induced QCD, $ABAB$ model, dually weighted graphs (see e.g. [20] and references therein). Basically, all these models can be written in terms of quadratic interaction models, either by definition or by using character expansions. Thus, they are closely related with the Ising model we shall describe below. The Ising model is given by the partition function

$$Z_N(t, c) = \int e^{-N\,\mathrm{tr}(V_t^1(A))-N\,\mathrm{tr}(V_t^2(B))-Nc\,\mathrm{tr}(AB)}\, d\mu_N(A)d\mu_N(B)$$

with V_t^1 and V_t^2 two polynomials of one real variable with coefficients depending on parameters t. By paragraph 2.2, if V_t^1, V_t^2 are convex, for small enough parameters (t, c), the free energy $\log Z_N(t)$ expands into a generating function for two-colored maps with vertices prescribed by V_t^1 and V_t^2. The interaction AB serves to generate edges between vertices of different colors. Thus, when $V_t^1(x) = V_t^2(x) = tx^4$, the model really looks like a generalization of the standard Ising model, with spins lying on a random quadrangulation rather than on $\mathbb{Z}^2$. Indeed, we have for small enough parameters t, c,

$$\frac{1}{N^2} \log Z_N(t, c)$$
$$= (1 - c^2)^{-1} \sum_{g=0}^{k} \frac{1}{N^{2g}} \sum_{k,\ell} \frac{1}{k!} \left(\frac{-t}{(1 - c^2)^2} \right)^k \frac{(-c)^\ell}{\ell!} C(k, \ell, g) + o(N^{-2k})$$

with $C(k, \ell, g)$ the number of maps with genus g with k vertices of valence 4 being assigned the sign $+1$ or -1, with exactly ℓ edges between vertices of different signs. This formula is reminiscent of the standard grand canonical partition function for the Ising model in $\mathbb{Z}^2$, where $C(k, \ell, g)$ is simply replaced by the number of configurations on a subset of $\mathbb{Z}^2$ rather than on random graphs. The genus is then related with the boundary conditions. In this case, where $V_t^1(x) = V_t^2(x) = tx^4$, an explicit formula for the limiting free energy was obtained in [31] from which important information such as phase transition could be derived [8] (these results were recovered in [7] by a purely combinatorial approach). For more general potentials, variational formulas generalizing those of the one matrix setting were obtained in [20]. A more detailed analysis of these limits is under study. The basic ingredient for these general potentials estimates is based on the remark that under μ_N, $A = UDU^*$ with U a unitary matrix following the Haar measure and D a diagonal matrix, independent of U. Therefore, the interaction in the Ising model is given by the spherical, or Itzykson–Zuber–Harish-Chandra, integral

$$I(D_1, D_2) = \int e^{Nc\,\mathrm{tr}(D_1 U D_2 U^*)}\, dU$$

with D_1 and D_2 the diagonal matrices of the eigenvalues of A, B and dU the Haar measure on the set of $N \times N$ unitary matrices. In [25], we obtained the large N asymptotics of spherical integrals. Namely, take a sequence of diagonal matrices (D_1^N, D_2^N) so that the empirical measures $N^{-1} \sum_{i=1}^N \delta_{D_j^N(ii)}$ converges weakly towards a probability measures μ_j for $j = 1, 2$. Then, if we set

$$I(\mu) = -\frac{1}{2} \iint \log |x - y| d\mu(x) d\mu(y) + \frac{1}{2} \int x^2 d\mu(x),$$

we have, if $I(\mu_1) < \infty$, $I(\mu_2) < \infty$,

$$\lim_{N \to \infty} \frac{1}{N^2} \log I(D_1^N, D_2^N)$$
$$= -\frac{1}{2} \inf \left\{ \int_0^1 \int u_t(x)^2 \rho_t(x) dx dt + \frac{\pi^2}{3} \int_0^1 \int \rho_t(x)^3 dx dt \right\} + I(\mu_1) + I(\mu_2).$$

The above infimum is taken over (ρ, u) on $(0, 1) \times \mathbb{R}$ so that $\nu_t(dx) = \rho_t(x)dx$ is a probability measure on $\mathbb{R}$ for all $t \in (0, 1)$, $t \to \nu_t$ is continuous with limit as t goes to zero (resp. one) given by μ_1 (resp. μ_2) and for all $t \in (0, 1)$, all $x \in \mathbb{R}$,

$$\partial_t \rho_t(x) + \partial_x(\rho_t(x)u_t(x)) = 0.$$

In [20] it was shown that the infimum is taken at a couple (ρ, u) so that $f = u + i\pi\rho$ satisfies the complex Burgers equation

$$\partial_t f_t(x) + f_t(x) \partial_x f_t(x) = 0.$$

These formulae are proved by large deviation estimates for N non-intersecting Brownian motions evaluated at extremely small time N^{-1}. Complex Burgers equation also appears in discrete analogous settings coming from tiling, see e.g. [27]. Spherical integrals are rather fundamental objects since they are related with the characters of the symmetric group; the above limits give asymptotics of Schur functions, cf. [21].

4. Conclusion: Matrix models input in combinatorics

Even though rather indirect, the matrix model approach to the enumeration of maps have proved to be powerful since it permits to consider quite general maps. For general type of vertices, the formulas obtained by this method are often not so much explicit, but this should be no surprise. When possible, the bijective approach provides more detailed information, such as the diameter of the graph with a given number of vertices (an information which was never grasped by the matrix model approach so far). The matrix model approach shows that some (maybe) unexpected tools can be used to solve these combinatorial problems; let us cite characters expansions (see [24] and references therein), Brownian motion and stochastic calculus (see [25]). We believe

also that the description of the generating functions $\mu_t(x^p)$ as the expectation under a probability measure, or tracial state, is a rather powerful remark. It allows to give some information on Tutte's solutions to the equations for generating functions of maps, seen as a solution to Schwinger–Dyson's equation (7). For instance, it was shown in [16] that the generating function $\tau(z - A)^{-1}$ for the Ising model with general polynomial potentials satisfies an algebraic equation.

This field has experienced quite a lot of developments in the last few years, attracting the interests of theoretical physicists and of mathematicians from diverse fields such as combinatorics, integrable systems or probability. Central in the problem of the enumeration of maps is the Schwinger–Dyson equation (7) which encodes most of the induction relations satisfied by the numbers of interest. The study of its solution is the heart of the problem of enumerating maps and, at least in the multi-matrix model, also attracted the attention of free probabilists. Indeed, in free probability, the reverse question (i.e. given a tracial state, find a potential V_t so that (7) is satisfied) serves to define the so-called conjugate variables which are central in free entropy questions. However, most issues in free probability are not related with small perturbative potentials as considered in this survey but on the contrary with very strong potentials. The understanding of matrix models is then extremly limited, since even the question of the convergence of the free energy is still unsettled.

References

[1] Albeverio S., Pastur L., Shcherbina, M., On the $1/n$ expansion for some unitary invariant ensembles of random matrices. *Comm. Math. Phys.* **224** (2001), 271–305.

[2] Ambjørn, J., Chekhov, L., Kristjansen, C. F., and Makeenko, Yu., Matrix model calculations beyond the spherical limit. *Nuclear Phys. B* **404** (1993), 127–172; Erratum ibid. **449** (1995), 681.

[3] Ben Arous, G., Guionnet, A., Large deviations for Wigner's law and Voiculescu's non-commutative entropy. *Probab. Theory Related Fields* **108** (1997), 517–542.

[4] Bender, E., Some unsolved problems in map enumeration. *Bull. Inst. Combin. Appl.* **3** (1991), 51–56.

[5] Bender, E., Canfield, E., The number of degree-restricted rooted maps on the sphere. *SIAM J. Discrete Math.* **7** (1994), 9–15.

[6] Bessis, D., Itzykson, C., Zuber, J. B., Quantum field theory techniques in graphical enumeration. *Adv. in Appl. Math.* **1** (1980), 109–157.

[7] Bousquet-Melou, M., Schaeffer, G., The degree distribution in bipartite planar maps: applications to the Ising model. Preprint, 2002; arXiv:math.CO/0211070.

[8] Boulatov, D., Kazakov, V., One-dimensional string theory with vortices as the upside-down matrix oscillator. *Internat. J. Modern Phys. A* **8** (1993), 809–851.

[9] Bouttier, J., Di Francesco, P., Guitter, E., Census of planar maps: from the one-matrix model solution to a combinatorial proof. *Nuclear Phys. B* **645** (2002), 477–499.

[10] Bouttier, J., Di Francesco, P., Guitter, E., Planar maps as labeled mobiles. *Electron. J. Combin.* **11** (2004), Research Paper 69, 27 pp. (electronic).

[11] Brézin, E., Itzykson, C., Parisi, G., and Zuber, J. B., Planar diagrams. *Comm. Math. Phys.* **59** (1978), 35–51.

[12] Chassaing, P., Schaeffer, G., Random planar lattices and integrated superBrownian excursion. *Probab. Theory Related Fields* **128** (2004), 161–212.

[13] Cori, R., Vauquelin, B., Planar maps are well labeled trees. *Canad. J. Math.* **33** (1981), 1023–1042.

[14] Deift, P., Kriecherbauer, T., McLaughlin, K. T.-R., New results on the equilibrium measure for logarithmic potentials in the presence of an external field. *J. Approx. Theory* **95** (1998), 388–475.

[15] Di Francesco, P., and Ginsparg, P., and Zinn-Justin, J., 2D gravity and random matrices. *Phys. Rep.* **254** (1995), 133pp.

[16] Eynard, B., Master loop equations, free energy and correlations for the chain of matrices. *J. High Energy Phys. A* **11** (2003), 018, 45 pp. (electronic).

[17] Ercolani, N. M., and McLaughlin, K. D. T.-R., Asymptotics of the partition function for random matrices via Riemann-Hilbert techniques and applications to graphical enumeration. *Internat. Math. Res. Notices* **14** (2003), 755–820.

[18] Giménez, O., Noy, M., Asymptotic enumeration and limit laws of planar graphs. Preprint, 2005; ariv:math.CO/0501269.

[19] Goulden, I. P., Jackson, D. M., *Combinatorial enumeration*. Wiley-Interscience Series in Discrete Mathematics, John Wiley & Sons Inc., New York 1983.

[20] Guionnet, A., First order asymptotics of matrix integrals; a rigorous approach towards the understanding of matrix models. *Comm. Math. Phys.* **244** (2004), 527–569.

[21] Guionnet, A., Large deviations and stochastic calculus for large random matrices. *Probab. Surv.* **1** (2004), 72–172.

[22] Guionnet, A., Maurel-Segala, E., Combinatorial aspects of matrix models. *Alea*, to appear 2006 (electronic).

[23] Guionnet, A., Maurel-Segala, E., Second order asymptotics for matrix models. Preprint, 2006; http://front.math.ucdavis.edu/math.PR/0601040.

[24] Guionnet, A., Maïda, M., Character expansion method for the first order asymptotics of a matrix integral. *Probab. Theory Related Fields* **132** (2005), 539–578.

[25] Guionnet, A., Zeitouni, O., Large deviations asymptotics for spherical integrals. *J. Funct. Anal.* **188** (2002), 461–515.

[26] Harary, F., Unsolved problems in the enumeration of graphs. *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **5** (1960), 63–95.

[27] Kenyon, R., Okounkov, A., Limit shapes and the complex Burgers equation. Preprint, 2005; http://front.math.ucdavis.edu/math.AG/0512573.

[28] Knizhik, V., Polyakov, A., and Zamolodchikov, A., Fractal structure of 2D-quantum gravity. *Mod. Phys. Lett A* **3** (1988), 819–826.

[29] Maurel-Segala, Edouard, High order expansion for matrix models. Preprint, 2006.

[30] Mehta, M.L., *Random matrices*. Third edition, Pure Appl. Math. (Amsterdam) 142, Elsevier/Academic Press, Amsterdam 2004.

[31] Mehta, M. L., A method of integration over matrix variables. *Comm. Math. Phys.* **79** (1981), 327–340.

[32] Schaeffer, G., Bijective census and random generation of Eulerian planar maps with pre-scribed vertex degrees. *Electron. J. Combin.* **4** (1997), Research Paper 20, 14 pp. (elec-tronic).

[33] 't Hooft, G., Magnetic monopoles in unified gauge theories. *Nuclear Phys. B* **79** (1974), 276–284.

[34] Tutte, W. T., A census of planar triangulations. *Canad. J. Math.* **14** (1962), 21–38.

[35] Tutte, W. T., A new branch of enumerative graph theory. *Bull. Amer. Math. Soc.* **68** (1962), 500–504.

[36] Zvonkin, A., Matrix integrals and map enumeration: an accessible introduction. *Math. Comput. Modelling* **26** (1997), 281–304.

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