

# The weak/strong survival transition on trees and nonamenable graphs

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**Abstract.** Various stochastic processes on nonamenable graphs and manifolds of exponential volume growth exhibit phases that do not occur in the corresponding processes on amenable graphs. Examples include: (1) *branching diffusion* and random walk on hyperbolic space, which for intermediate branching rates may survive globally but not locally; (2) *contact processes* on homogeneous trees, which likewise can survive globally while dying out locally; and (3) *percolation* on Cayley graphs of nonamenable groups, where for certain parameter values infinitely many infinite percolation clusters may coincide. This article surveys some of what is known about the intermediate phases and the upper phase transitions for these processes.

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## 1. Branching Brownian motion and random walk

**B.B.M. in the hyperbolic plane.** Branching Brownian motion in the hyperbolic plane  $\mathbb{H}$  is perhaps the simplest process exhibiting the weak/strong survival transition. It evolves as follows: At time 0, a single particle located at a specific point  $x_0 \in \mathbb{H}$  begins a Brownian motion. At random exponentially distributed times with mean  $1/\lambda$ , independent of the motion, the particle undergoes *binary fission*, in which a replicate particle is created at the current location of the fissioning particle. The offspring particles behave as their parents, executing Brownian motions from the places of their births and undergoing further binary fissions at exponentially distributed random times; their behavior is completely independent of their parents' and other particles' behaviors, except for the locations of their births.

The behavior of the branching Brownian motion is controlled by the fission parameter  $\lambda$ . The size  $N_t$  of the population at times  $t$  is a simple continuous-time Galton–Watson process with  $EN_t = e^{\lambda t}$ . If a particle is chosen at random from the  $N_t$  particles in existence at time  $t$ , the distribution of its position has as its density the heat kernel  $p_t(x, \cdot)$ . This is known to behave asymptotically as  $t \rightarrow \infty$  like

$$p_t(x, y) \sim C_{x,y} t^{-3/2} \exp\{-t/8\} \quad (1)$$

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for constants  $0 < C_{x,y} < \infty$  varying smoothly with  $x, y$ . Thus, the mean number of particles located in a bounded neighborhood  $U$  of  $y$  grows/decays roughly as  $\exp\{(\lambda - 1/8)t\}$ . It is not difficult to deduce the following.

**Proposition 1.** *For  $\lambda \leq 1/8$ , branching Brownian motion survives weakly, that is, for every bounded region  $U$  the number of particles located in  $U$  is eventually 0, w.p.1. For  $\lambda > 1/8$  it survives strongly, that is, for every open set  $U$  the number of particles located in  $U$  converges to  $\infty$  as  $t \rightarrow \infty$ , w.p.1.*

In the weak survival phase, all particle trajectories tend to the boundary circle  $\partial\mathbb{H}$  of the hyperbolic plane. Define  $\Lambda$  to be the set of all accumulation points of particle trajectories in  $\partial\mathbb{H}$ . With probability 1,  $\Lambda$  is a nonempty, compact subset of  $\partial\mathbb{H}$ .

**Theorem 2** ([14]). *For  $\lambda \in (0, 1/8]$ , the limit set  $\Lambda$  is, with probability 1, a Cantor set of Hausdorff dimension*

$$\delta = \delta(\lambda) = \frac{1}{2}(1 - \sqrt{1 - 8\lambda}). \quad (2)$$

Observe that as  $\lambda \rightarrow 1/8$  from below, the Hausdorff dimension approaches  $1/2$ , not 1 (the dimension of the ambient boundary  $\partial\mathbb{H}$ ), as one might at first suspect. In the strong survival phase  $\lambda > 1/8$  the limit set  $\Lambda$  is the entire boundary  $\partial\mathbb{H}$ , so Theorem 2 shows that the Hausdorff dimension behaves discontinuously at the critical parameter  $\lambda = 1/8$ . Moreover, it shows that the *critical exponent* for  $\delta(\lambda)$  at the transition is  $1/2$ . This, as it turns out, is closely related to the exponent  $3/2$  appearing in the asymptotic formula (1) for the heat kernel.

Theorem 2 has been generalized to branching Brownian motion and certain other isotropic branching random walks on higher-dimensional hyperbolic spaces  $\mathbb{H}^d$  by Karpelevich, Pechersky, and Suhov [8]: they prove that, for branching Brownian motion in  $\mathbb{H}^d$ , the Hausdorff dimension  $\delta(\lambda)$  of the limit set  $\Lambda$  converges up to  $(d-1)/2$  as  $\lambda$  approaches the critical point from below.

**Branching random walk on  $\mathbb{T}^d$ .** The existence of a weak survival phase for B.B.M. in  $\mathbb{H}$  is a consequence of the exponential decay (1) of the heat kernel. A fundamental theorem of Kesten [9], [10] asserts that exponential decay of random walk transition probabilities is characteristic of nonamenable groups. Thus, branching random walk in any nonamenable group must also have a weak survival phase.

The transition from weak survival to strong survival is understood only for branching random walk on the homogeneous tree  $\mathbb{T}^d$  of degree  $d \geq 3$  (the Cayley graph of the free product  $\Gamma^d := (\mathbb{Z}_2)^{*d}$ ). Let  $\{p_i\}_{i \in A \cup \{1\}}$  be a positive probability distribution on  $A \cup \{1\}$  where  $A$  is the set of generators of  $\Gamma^d$ , and denote by  $p_n(x, y)$  the  $n$ -step transition probabilities of the random walk with step distribution  $\{p_i\}$ . The branching random walk associated with the probability distribution  $\{p_i\}$  is constructed as follows: At time  $n = 0$ , the process is initiated by a single particle located at the site 1 (the root of the tree). The population  $X_{n+1}$  of each subsequent generation  $n + 1$

is obtained from  $X_n$  in two steps: First, each particle  $\zeta$  in  $X_n$  reproduces, creating a random number  $N_\zeta \geq 1$  of replica particles, all located at the same vertex of  $\mathbb{T}^d$  as  $\zeta$ . The distribution of the offspring count  $N_\zeta$  is geometric+1 with mean  $\lambda > 1$ . Second, each particle moves to a randomly chosen neighboring vertex, according to the distribution  $\{p_i\}$ .

Let  $R > 1$  be the spectral radius of the base random walk, that is,

$$R^{-1} := \lim_{n \rightarrow \infty} p_n(x, y)^{1/n}. \tag{3}$$

Just as for B.B.M. in  $\mathbb{H}$ , if the mean offspring number  $\lambda$  exceeds  $R$  then the number of particles located at the root vertex 1 will explode almost surely. However, if  $\lambda \leq R$ , then the branching random walk survives only weakly: although the total population size grows exponentially at rate  $\lambda$ , the number of particles located at any particular vertex will eventually be zero, w.p.1. (That the B.R.W. survives only weakly at the critical point  $\lambda = R$  follows because the base random walk is  $R$ -transient: see [24].) Thus, for  $\lambda \leq R$ , particle trajectories converge to the space  $\partial\mathbb{T}^d$  of ends of the tree. Hence, we may define

$$\Lambda := \{\text{ends in which the BRW survives}\}. \tag{4}$$

The Hausdorff dimension  $\delta_H(\Lambda)$ , computed with respect to the natural metric<sup>1</sup> on  $\partial\mathbb{T}^d$ , is a natural measure of the growth of the B.R.W. in the weak survival phase, for the following reason: If  $M_m$  is the number of vertices of  $\mathbb{T}^d$  at distance  $m$  from the root that are ever visited by particles of the B.R.W., then with probability one,

$$\lim_{m \rightarrow \infty} M_m^{1/m} = \theta(\lambda) \tag{5}$$

where [7],

$$\delta_H(\Lambda) = \frac{\log \theta(\lambda)}{\log 2}. \tag{6}$$

Denote by  $G_x(\lambda)$  the Green's function and  $F_x(\lambda)$  the first-passage generating function of the base random walk, that is,

$$G_x(\lambda) = \sum_{n=0}^{\infty} p_n(1, x), \tag{7}$$

$$F_x(\lambda) = G_x(\lambda)/G_1(\lambda).$$

**Theorem 3** ([7]). *The Malthusian parameter  $\theta(\lambda)$  is the unique positive number such that*

$$\sum_{i \in A} \frac{F_i(\lambda)}{F_i(\lambda) + \theta(\lambda)} = 1. \tag{8}$$

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<sup>1</sup>The natural metric  $d$  is defined by  $d(\alpha, \beta) = 2^{-N(\alpha, \beta)}$ , where  $N(\alpha, \beta)$  denotes the number of common edges in the geodesic segments from the root to  $\alpha$  and from the root to  $\beta$ .

This parameter has critical exponent  $1/2$  at the critical point  $\lambda = R$ : that is, there exists a constant  $C > 0$  such that as  $\lambda \rightarrow R$  from below,

$$\theta(R) - \theta(\lambda) \sim C\sqrt{R - \lambda}. \quad (9)$$

Furthermore,

$$\theta(\lambda) \leq \sqrt{d - 1} \quad (10)$$

and equality holds if and only if the step distribution  $\{p_i\}$  is isotropic.

The formula (8) makes it clear that the critical exponent  $1/2$  in (9) is related to the exponent  $3/2$  occurring in the power law

$$p_n(1, x) \sim C_x R^{-n} n^{-3/2} \quad (11)$$

for the base random walk transition probabilities [24]. This is because (11) is determined by the singularity of the Green's function  $G_x(\lambda)$  at  $\lambda = R$ , by standard Tauberian theorems, and this in turn has the same singular asymptotics as the first-passage generating functions  $F_x(\lambda)$ . It is conjectured that the local limit theorem (11) holds more generally for random walks on nonelementary Fuchsian groups (discrete groups of isometries of the hyperbolic plane), but this has been proved only for Fuchsian groups containing free groups as subgroups of finite index. It is natural to expect that branching random walks on such groups will have weak/strong survival transitions of the same type as on homogeneous trees.

## 2. Contact processes

**Weak survival.** Let  $G = (V, E)$  be the Cayley graph of a finitely generated group  $\Gamma$ , with edges labeled by elements of the generating set  $A$ , and let  $\mathcal{P} = \{p_a\}_{a \in A}$  be a probability distribution on  $A$ . The *contact process* with intensity parameter  $\lambda > 0$  and infection rates  $\mathcal{P}$  is a Markov process on the configuration space  $\{0, 1\}^V$  (here  $0 =$  "healthy" and  $1 =$  "infected") that evolves as follows: (A) Infected vertices "recover" (become healthy) at rate 1. (B) Healthy vertices  $x$  become infected at rate

$$\lambda \sum_{j \in A: xj \in \xi_t} p_j$$

where  $\xi_t$  is the set of vertices that are infected at time  $t$ . If the probability distribution  $\mathcal{P}$  is uniform on  $A$ , the contact process is said to be *isotropic*. If the generating set  $A$  and the rates  $\mathcal{P}$  are symmetric (that is,  $a \in A$  implies  $a^{-1} \in A$  and  $p_a = p_{a^{-1}}$ ) then the contact process is said to be *symmetric*. The default initial condition is  $\xi_0 = \{1\}$ , where  $1$  denotes the group identity. The contact process is said to survive weakly if  $\xi_t \neq \emptyset$  for all  $t > 0$  but  $\xi_t \cap F = \emptyset$  eventually for every finite set  $F \subset V$ . It survives strongly if for every nonempty set  $F \subset V$ , the intersection  $\xi_t \cap F$  is nonempty at indefinitely large times  $t$ .

Most of what is known about existence of weak and strong survival phases for contact processes is restricted to groups  $\Gamma$  whose associated Cayley graphs  $G$  are trees.

**Theorem 4** ([20], [18], [23], [25]). *Assume that  $G$  is the homogeneous tree  $\mathbb{T}^d$  of degree  $d \geq 3$ . Then for the isotropic contact process there exist constants  $0 < \lambda_c < \lambda_u < \infty$  (depending on  $d$ ) so that*

- (a)  $\lambda \leq \lambda_c \implies$  *ultimate extinction with probability 1.*
- (b)  $\lambda_c < \lambda \leq \lambda_u \implies$  *weak survival with positive probability.*
- (c)  $\lambda > \lambda_u \implies$  *strong survival with positive probability.*

Unlike the corresponding results for branching random walks and branching Brownian motion on hyperbolic spaces, Theorem 4 is surprisingly difficult (at least for small  $d$ ). The proofs in [20], [18], and [23] rely heavily on both isotropy and the absence of cycles in the graph. The following result weakens the isotropy requirement.

**Theorem 5** ([16]). *Assume that  $G$  is a homogeneous tree of degree  $d \geq 3$ . Assume further that the rates  $\mathcal{P}$  are symmetric and weakly isotropic in the sense that there are generators  $a, b$  with  $a \neq b^{\pm 1}$  so that  $p_a = p_b$ . Then the contact process with rates  $\mathcal{P}$  has a weak survival phase, that is, there exist  $0 < \lambda_c < \lambda_u < \infty$  so that conclusions (a), (b), and (c) of Theorem 4 are valid.*

Alan Stacey (unpublished) has recently shown that the weak isotropy hypothesis is unnecessary.

**Size of the limit set.** Consider now the isotropic contact process on a homogeneous tree  $\mathbb{T}^d$  of degree  $d \geq 3$ . By Theorem 4, there is a nontrivial weak survival phase  $\lambda_c < \lambda < \lambda_u$ . By definition of weak survival, any finite set of vertices must eventually be vacated, with probability 1. Therefore, the set of occupied vertices must recede to  $\partial\mathbb{T}^d$  as  $t \rightarrow \infty$ . As for branching random walk in  $\mathbb{T}^d$ , define the *limit set*  $\Lambda$  to be the set of ends in which the contact process survives.

**Theorem 6** ([15]). *For the isotropic contact process in the weak survival phase, the Hausdorff dimension  $\delta_H(\Lambda)$  of the limit set  $\Lambda$  is a.s. constant on the event of survival, and satisfies the inequality*

$$\delta_H(\Lambda) \leq \frac{1}{2} \delta_H(\partial\mathbb{T}^d). \tag{12}$$

*Furthermore,  $\delta_H(\Lambda)$  is continuous [22] and strictly increasing [12] in the parameter  $\lambda$ , and equality in (12) holds at  $\lambda = \lambda_u$ .*

The inequality (12) holds for essentially the same reason as for branching random walk: If  $\delta_H(\Lambda)$  were greater than  $(1/2)\delta_H(\partial\mathbb{T}^d)$  then there would be particle trajectories extending from the root vertex to vertices far from the root and then back to

the root, contradicting weak survival. As for branching random walk on a homogeneous tree, the inequality (12) remains valid for nonisotropic but *symmetric* contact processes.

The Hausdorff dimension of the limit set  $\Lambda$  is simply related to a hitting probability associated to the contact process. Let  $x \in V$  be a vertex at distance  $n$  from the root vertex 1, and define

$$u_n = P\{x \in \xi_t \text{ for some } t > 0\} \quad (13)$$

to be the probability that the vertex  $x$  is ever infected. By isotropy, this probability is the same for all vertices  $x$  at distance  $n$  from 1. It is apparent that  $u_{m+n} \geq u_m u_n$ , and so

$$\lim_{n \rightarrow \infty} u_n^{1/n} := \beta = \beta(\lambda) \quad (14)$$

exists and is  $\leq 1$ .

**Theorem 7** ([15]).

$$\delta_H(\Lambda) = -\frac{\log(d-1)\beta}{\log 2}. \quad (15)$$

**Critical exponent.** For the isotropic contact process on the tree  $\mathbb{T}^d$  of degree  $d \geq 3$ , the Hausdorff dimension  $\delta(\lambda) := \delta_H(\Lambda)$  varies continuously with the intensity  $\lambda$  for  $\lambda \leq \lambda_u$  [22], and increases *strictly* with  $\lambda$  in the interval  $\{\lambda : \delta(\lambda) < 1/2\delta_H(\partial\mathbb{T}^d)\}$  [12]. Define

$$\lambda_* = \sup\{\lambda : \delta(\lambda) < 1/2\delta_H(\partial\mathbb{T}^d)\}. \quad (16)$$

**Conjecture 8.**  $\lambda_* = \lambda_u$ .

Recall that the critical exponent at the upper critical point for branching random walk on  $\mathbb{T}^d$  is  $1/2$ , by formula (9). It is believed that the phase transition for the contact process is of the same type as the corresponding phase transition for branching random walk, and so it is natural to conjecture that the critical exponent is again  $1/2$ :

**Conjecture 9.**

$$\lim_{\lambda \rightarrow \lambda_*} \frac{\log(\delta_H(\partial\mathbb{T}^d) - 2\delta(\lambda))}{\lambda_* - \lambda} = \frac{1}{2}.$$

Further evidence for the truth of this conjecture is provided by the main result of [17], which we now describe. Consider the isotropic contact process  $\xi_t$  on  $\mathbb{T}^d$  in the weak survival phase  $\lambda \in (\lambda_c, \lambda_u]$ . For any site  $x$ , the total infection time

$$J(x) := \int_0^\infty \mathbf{1}\{x \in \xi_t\} dt \quad (17)$$

is finite with probability 1. It is known [12] that if  $\lambda < \lambda_*$  then  $P\{x \in \xi_t\}$  is exponentially decaying in  $t$ , and so  $EJ(x) < \infty$ . Because the hitting probability  $u_n$  decays exponentially in  $n$  even at the critical point  $\lambda = \lambda_*$ , it is natural to expect that the conditional expectation of  $J(x)$  given  $J(x) > 0$  is finite.

**Conjecture 10.** There exists a constant  $C = C_d$  depending only on the degree  $d$  of the tree  $\mathbb{T}^d$  such that, for every vertex  $x$  and all  $\lambda \leq \lambda_*$ ,

$$E(J(x) | J(x) > 0) \leq C. \quad (18)$$

The analogous statement is known to be true for branching random walk. For the contact process, it is at least as plausible as Conjecture 9.

**Theorem 11** ([17]). *If Conjecture 10 is true then there is a finite constant  $C = C_d$  so that for all  $\lambda < \lambda_*$ ,*

$$\frac{1}{2} \delta_H(\partial \mathbb{T}^d) - \delta(\lambda) \leq C \sqrt{\lambda_* - \lambda}. \quad (19)$$

Thus, if Conjecture 10 is true, and if there is a critical exponent, then it cannot be less than  $1/2$ . The proof of Theorem 11 in [17] also suggests that  $1/2$  is the correct value, as the inequalities in the proof are very likely approximate equalities.

### 3. Percolation

**Coexistence of infinite clusters.** In *Bernoulli site (resp., bond) percolation* on a graph  $G$ , vertices (resp., edges) are colored blue or red independently, blue with probability  $p$ , red with probability  $1 - p$ . For brevity we shall discuss only site percolation; however, most of the results and conjectures have natural analogues for bond percolation.

In site percolation, interest focuses on the connected clusters of blue vertices, and in particular on the existence/uniqueness and geometry of infinite blue clusters. *Percolation* is said to occur if there is an infinite blue cluster. For any infinite graph there exists a unique threshold  $p_c \in [0, 1]$  for the Bernoulli parameter  $p$  above which percolation occurs with positive probability, and below which it occurs with probability zero. Burton and Keane [5] showed that if the ambient graph  $G$  is the Cayley graph of a finitely generated, amenable group, then infinite blue clusters, if they exist, are unique w.p.1. Grimmett and Newman [6] showed that uniqueness of infinite clusters need not hold in nonamenable graphs: in particular, they showed that Bernoulli percolation on  $\mathbb{Z} \times \mathbb{T}^d$  has infinitely many infinite clusters for certain values of  $p$ , provided the degree  $d$  is sufficiently large.

A graph is called *transitive* if its automorphism group acts transitively on the vertex set, and is called *nonamenable* if there exists a constant  $\varepsilon > 0$  such that for any finite set  $V_0$  of vertices,  $|\partial V_0| > \varepsilon |V_0|$ . (Here  $\partial V_0$  denotes the set of vertices not in  $V_0$  that are connected by edges to vertices of  $V_0$ .)

**Conjecture 12** ([1]). If  $G$  is a transitive, nonamenable graph, then there exists a nonempty interval  $I = (p_c, p_u)$  such that for all  $p \in I$ , Bernoulli- $p$  site percolation has infinitely many infinite blue clusters.

In full generality, this remains unresolved. However, two important results have been obtained:

**Theorem 13** ([2]). *Let  $G$  be a transitive, nonamenable, planar graph with one end. Then there exist constants  $0 < p_c < p_u < 1$  such that*

- (a)  $p \leq p_c \implies$  *no infinite blue clusters;*
- (b)  $p_c < p < p_u \implies$  *infinitely many infinite blue clusters;*
- (c)  $p_u \leq p \implies$  *one infinite blue cluster.*

A connected, transitive graph is said to have one end if the subgraph obtained by deleting any finite set of vertices remains connected. Observe that the theorem asserts uniqueness of the infinite cluster at the upper transition point  $p_u$ : this contrasts with the analogous transition on  $\mathbb{Z} \times \mathbb{T}^d$ , where it is known [21] that at least for large  $d$  there are infinitely many infinite clusters at  $p_u$ . See [3], [4] for discussion of related issues.

**Theorem 14** ([19]). *For every finitely generated, nonamenable group there is a Cayley graph for which Conjecture 12 is true.*

**Percolation clusters in hyperbolic tessellations.** A *Fuchsian group* is a discrete group of isometries of the hyperbolic plane  $\mathbb{H}$ . Let  $\Gamma$  be a co-compact Fuchsian group with finite, symmetric generating set  $A$ , and let  $G$  be the Cayley graph. Then  $G$  may be naturally embedded in  $\mathbb{H}$  in such a way that edges are geodesic segments in  $\mathbb{H}$ , and so that any compact subset of  $\mathbb{H}$  contains only finitely many vertices of  $G$ . Denote by  $x_0 \in \mathbb{H}$  the vertex of  $G$  corresponding to the group identity  $1 \in \Gamma$ .

Consider Bernoulli- $p$  site percolation on  $G$ , and let  $K$  be the connected blue cluster containing  $x_0$ . On the event that  $K$  is infinite, vertices in  $K$  will accumulate at the boundary circle  $\partial\mathbb{H}$ : define  $\Lambda$  to be the (closed) set of accumulation points.

**Theorem 15** ([11]). *If  $p_c < p < p_u$  then on the event that  $K$  is infinite the limit set  $\Lambda$  is a Cantor set of Lebesgue measure 0.*

Recall that for the contact process on a homogeneous tree, the Hausdorff dimension of the limit set is discontinuous at the transition from weak to strong survival. It is natural to ask how the Hausdorff dimension of the limit set  $\Lambda$  of a percolation cluster in  $\mathbb{H}$  behaves as  $p \rightarrow p_u$  from below.

**Theorem 16** ([13]). *For each  $p \in (p_c, p_u)$  the Hausdorff dimension  $\delta(p)$  of the limit set  $\Lambda$  in Bernoulli- $p$  site percolation is almost surely constant. The function  $p \mapsto \delta(p)$  is continuous and strictly increasing in  $p$ , with limit 1 as  $p \rightarrow p_u$ .*

Thus, the nature of the phase transition for Bernoulli percolation seems to be different from that for contact processes and branching random walks.

The use of  $\delta(p)$  as a measure of the size of percolation clusters is not unreasonable for Fuchsian hyperbolic groups, but a more natural measure might be the volume growth rate

$$\varrho = \varrho(p) = \lim_{R \rightarrow \infty} R^{-1} \log \text{card} K_R \tag{20}$$

where  $K_R$  denotes the intersection of the cluster  $K$  with the ball of radius  $R$  centered at the root  $x_0$ . (For Fuchsian groups, it can be shown that  $\varrho(p) = \delta(p)$ .) Whereas  $\delta(p)$  depends for its definition on the existence of a geometric boundary of the ambient space,  $\varrho(p)$  can in principle be used for an arbitrary infinite group, using the graph metric to measure volume growth (existence of the limit must be proved, of course).

**Conjecture 17.** Assume that  $G$  is a transitive, nonamenable graph with nonempty coexistence phase  $(p_c, p_u)$ . If there is a unique infinite cluster a.s. at  $p = p_u$ , then  $\varrho(p)$  converges, as  $p \rightarrow p_u$  from below, to the volume growth rate of the ambient graph  $G$ .

The proof of Theorem 16 leads to an interesting variational formula for the Hausdorff dimension  $\delta(p)$  of percolation clusters in co-compact Fuchsian groups. Define the *connectivity function*  $\tau : \Gamma \rightarrow [0, 1]$  as follows:

$$\tau(x) = \tau(x; p) = P_p\{x \in K\}. \tag{21}$$

By the FKG inequality,  $\tau$  satisfies a log-subadditivity inequality on  $\Gamma$ :

$$\tau(xy) \geq \tau(x)\tau(y). \tag{22}$$

The function  $\tau$  may be extended to a function on the entire hyperbolic plane  $\mathbb{H}$  by setting  $\tau(w) = \tau(x)$  where  $x$  is the vertex of the Cayley graph  $G$  nearest  $w$  (with some convention for ties). Now consider the geodesic flow  $\Phi_t$  on the unit tangent bundle of  $\mathbb{H}/\Gamma$ : geodesics may be lifted to  $\mathbb{H}$ , and so by (21) the log-connectivity function evaluated along geodesics is subadditive for the geodesic flow. Therefore, by Kingman’s subadditive ergodic theorem, for any ergodic, invariant probability measure  $\mu$  for the flow  $\Phi_t$ , there exists a constant  $\beta(\mu) = \beta(\mu; p)$  so that  $\mu$ -almost surely, the connectivity function  $\tau$  decays at rate  $\beta(\mu)$  along geodesics, that is,

$$\lim_{t \rightarrow \infty} t^{-1} \log \tau \circ \Phi_t = \beta(\mu). \tag{23}$$

Denote by  $\mathcal{I}$  the set of all invariant probability measures for the geodesic flow, and by  $h(\mu)$  the Kolmogorov–Sinai entropy of the geodesic flow relative to the invariant measure  $\mu$ .

**Theorem 18** ([13]). *The decay rate function is jointly continuous in  $\mu$  and  $p$ , and for each  $\mu$  is strictly increasing in  $p$  for  $p \in (p_c, p_u)$ . Moreover,  $\beta(\mu; p) = 0$  for all  $\mu \in \mathcal{I}$  and all  $p \geq p_u$ . For every  $p$  the Hausdorff dimension  $\delta(p)$  of the limit set  $\Lambda$  is,  $P_p$ -almost surely on the event  $|K| = \infty$ , given by*

$$\delta(p) = \max_{\mu \in \mathcal{I}} (h(\mu) + \beta(\mu; p)). \tag{24}$$

Although it is by no means obvious, this is the natural analogue of formula (8) for the H.D. of the limit set of a branching random walk on  $\mathbb{T}^d$ . Formula (24) has further implications for the geometry of percolation clusters. Recall that  $K_R$  is the intersection of the percolation cluster  $K$  with the (hyperbolic) ball of radius  $R$  centered at  $x_0$ . The cardinality of  $K_R$  grows exponentially, at rate  $\delta(p)$  (see (20)), and most of the vertices in  $K_R$  are at distance nearly  $R$  from  $x_0$ . Suppose that one of these is chosen at random: then the geodesic ray from  $x_0$  through the randomly chosen vertex of  $K_R$  will be approximately  $\mu$ -generic, where  $\mu$  is the maximizing measure in (24).

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