New developments in stochastic dynamics

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Abstract. Flows of random coalescing maps or flows of random transition probabilities can arise from simple stochastic differential equations when Ito's theory of strong solutions ceases to apply.

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Introduction

A stationary motion on the real line with independent increments is described by a Levy process, or equivalently by a convolution semigroup of probability measures. This naturally extends to "rigid" motions represented by Levy processes on Lie groups. If one assumes the continuity of the paths, a convolution semigroup on a Lie group G is determined by an element of the Lie algebra $\mathfrak{g}$ (the drift) and a scalar product on $\mathfrak{g}$ (the diffusion matrix). We call them the local characteristics of the convolution semigroup. We will be interested in stationary "fluid" random evolutions which have independent increments. They can be modelled by stochastic differential equations driven by Wiener processes. These have been studied for more than fifty years. In particular, it was shown that stochastic flows driven by smooth Brownian vector fields on a compact manifold define flows of diffeomorphisms. This is also true on non compact manifolds under appropriate conditions of non explosion ([7]). Such flows can be viewed as infinitely divisible limits of products of i.i.d. (independant and indentically distributed) random diffeomorphisms, and the theory is at least formally very similar to the theory of Brownian motion on Lie groups ([14], [19], [12]). Their laws can be viewed as convolution semigroups of probability measures on the group of diffeomorphisms. They are characterized by two functions: the covariance of the Brownian vector field, or equivalently its auto reproducing space, which plays the role of a metric on the Lie algebra and a drift vector field. One can similarly extend the notion of Levy processes by introducing Poisson measures on the group itself. A remarkable result of Tsirelson (cf. [24]) shows that there is essentially no other way to define a process X_t with stationary independent increments on the unitary group of the Hilbert space. The *noise* defined by the increments of the flow, i.e. the family of σ-fields $\mathcal{F}_{s,t} = \sigma\{X_v X_u^{-1}, s \le u \le v \le t\}$ is classical, i.e. generated by additive

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increments of Wiener and (or) Poisson processes. But in recent years it appeared that this picture was not complete: indeed flows of non invertible transformations, and as well of transition probabilities, appear to play an important role in the theory.

To be more precise, on a compact manifold let $V_0, V_1, \dots, V_n$ be vector fields and $B^1, \dots, B^n$ be independent Brownian motions. Consider the SDE

$$dX_t = \sum_{k=1}^{n} V_k(X_t) \circ dB_t^k + V_0(X_t)\, dt, \tag{1}$$

which equivalently can be written as

$$df(X_t) = \sum_{k=1}^{n} V_k f(X_t) dB_t^k + \frac{1}{2} Af(X_t) dt \tag{2}$$

for every smooth function f and $Af = \sum_{k=1}^{n} V_k(V_k f) + V_0 f$. Observe that $Af^2 - 2fAf = \sum_{k=1}^{n} (V_k f)^2$. Then strong solutions of this SDE produce a flow of maps φ_t such that, for every x, $\varphi_t(x)$ is a strong solution of the SDE with $\varphi_0(x) = x$. When the vector fields are smooth, strong solutions are known to exist and to be unique. The framework can be extended to include flows of maps driven by vector field valued Brownian motions, which means essentially that $n = \infty$ (see for example [3], [12], [14], [19], [23]).

In a joint paper ([15]) with Olivier Raimond this was extended again to include flows of Markovian operators S_t which are solutions of the SPDE

$$dS_t f = \sum_{k=1}^{\infty} S_t(V_k f) dB_t^k + \frac{1}{2} S_t(Af) dt, \tag{3}$$

assuming that the covariance function $C = \sum_{k=1}^{\infty} V_k \otimes V_k$ of the Brownian vector field $\sum_{k=1}^{\infty} V_k B^k$ is compatible with A, namely that

$$Af^2 - 2fAf \geq \sum_{k=1}^{\infty} (V_k f)^2. \tag{4}$$

Existence and uniqueness of a flow of Markovian operators S_t, which is a Wiener solution of the previous SPDE in the sense that S_t is a function of the Brownian paths $(B^i)_{i \geq 1}$ up to time t, hold under rather weak assumptions.

The local characteristics of these flows are given by A and the covariance function C, and they determine the SDE or the SPDE. Under Lipschitz conditions we actually get strong solutions of stochastic differential equations. These solutions are of a regular type, namely:

(a) The probability that two points thrown in the fluid at the same time and at distance ε, separate at distance one in one unit of time tends to 0 as ε tends to 0.

(b) Such points will never hit each other.

But it was shown in [15] that covariance functions which are not smooth on the diagonal (e.g. covariance associated with Sobolev norms of order between $d/2$ and $(d+2)/2$, d being the dimension of the space) can produce Wiener solutions which define random evolutions of different types:

- turbulent evolutions where (a) is not satisfied, which means that two points thrown initially at the same place separate, even when there is no pure diffusion, i.e. that $Af^2 - 2fAf = \sum_{k=1}^{\infty}(V_k f)^2$;
- coalescing evolutions where (b) does not hold.

That paper was motivated by the works of physicists working on the Kraichnan model for turbulent advection (cf., for example, [9], [10], [4], [6]).

In a subsequent paper [16] we adopted a more general approach based on consistent systems of n-point Markovian Feller semigroups which can be viewed as determining the law of the motion of n indivisible points thrown into the fluid. Regular and coalescing evolutions are represented by flows of maps, and turbulent evolutions by flows of probability kernels $K_{s,t}(x, dy)$ describing how a point mass (made of a continuum of indivisible points) in x at time s is spread at time t. (Note that in this case, the motion of an indivisible point is not fully determined by the flow.)

Among turbulent evolutions, we can distinguish the intermediate ones where two points thrown in the fluid at the same place separate but can meet later, i.e. where (a) and (b) are both not satisfied. These flows can always be coupled with a coalescent flow.

Let us explain in more detail the contents of the paper. We give in the first section construction results from [16], which generalize a theorem by De Finetti on exchangeable variables. A stochastic flow of kernels K is associated with a general compatible family $(\mathsf{P}_t^{(n)},\ n \geq 1)$ of Feller semigroups. The flow K is induced by a flow of measurable mappings when

$$\mathsf{P}_t^{(2)} f^{\otimes 2}(x, x) = \mathsf{P}_t f^2(x),$$

for all $f \in C(M)$, $x \in M$ and $t \geq 0$. The Markov process associated with $\mathsf{P}_t^{(n)}$ represents the motion of n indivisible points thrown in the fluid. The key point is that the two notions are shown to be equivalent: the law of a stochastic flow of kernels is uniquely determined by the compatible system of n-point motions.

In Section 2 we define the noise associated with a flow and recall the notion of "black noise" introduced by Tsirelson.

Coalescing flows are defined in Section 3. A coalescing flow can be obtained from any flow of kernels the two-point motion of which hits the diagonal. Then the original flow is recovered by filtering the coalescing flow with respect to a sub-noise.

We give the example of Arratia's flow and consider briefly sticky flows.

In Section 4, we present the result of [18] in which the classification of solutions of Tanaka's equation is given.

Finally, in Section 5, we consider stochastic flows on the circle defined by SDE's driven by the white noise W, which exhibit most of the features of more general isotropic flows considered in [15] and [16].

1. Flows and their construction

This first section is rather formal since we chose to give a precise result. Its intuitive content is rather simple: flows of maps, and more generally flows of transition kernels, are described by their moments which are Markovian semigroups describing the motion of any finite number of points transported by the flow. We refer to [24] for an alternative approach to this construction.

1.1. Flows of maps. Let M be a compact separable metric space.

Definition 1.1.1. Let $(\mathsf{P}_t^{(n)},\ n \geq 1)$ be a family of Feller semigroups[1], defined on M^n and acting on $C(M^n)$, respectively. We say that this family is *consistent* as soon as for all $k \leq n$,

$$\mathsf{P}_t^{(k)} f(x_1, \ldots, x_k) = \mathsf{P}_t^{(n)} g(y_1, \ldots, y_n), \tag{1.1}$$

where f and g are any continuous functions such that

$$g(y_1, \ldots, y_n) = f(y_{i_1}, \ldots, y_{i_k}) \tag{1.2}$$

with $\{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$ and $(x_1, \ldots, x_k) = (y_{i_1}, \ldots, y_{i_k})$. We will denote by $\mathsf{P}^{(n)}_{(x_1,\ldots,x_n)}$ the law of the Markov process associated with $\mathsf{P}_t^{(n)}$ starting from $(x_1, \ldots, x_n)$.

This Markov process will be called the n-point motion (see also [21]).

We equip M with its Borel σ-field $\mathcal{B}(M)$. Let $(F, \mathcal{F})$ be the space of measurable mappings on M equipped with the σ-field generated by the evaluations at x for all x in M.

Definition 1.1.2. A probability measure Q on $(F, \mathcal{F})$ is called *regular* if there exists a measurable mapping $\mathcal{J} \colon (F, \mathcal{F}) \to (F, \mathcal{F})$ such that

$$(M \times F, \mathcal{B}(M) \otimes \mathcal{F}) \to (M, \mathcal{B}(M)),$$
$$(x, \varphi) \mapsto \mathcal{J}(\varphi)(x)$$

[1] $\mathsf{P}_t^{(n)}$ is a Feller semigroup on M^n if and only if $\mathsf{P}_t^{(n)}$ is positive (i.e. $\mathsf{P}_t^{(n)} f \geq 0$ for every $f \geq 0$), $\mathsf{P}_t^{(n)} 1 = 1$ and for every continuous function f, $\mathsf{P}_t^{(n)} f$ is continuous and $\lim_{t\to 0} \mathsf{P}_t^{(n)} f(x) = f(x)$, which implies the uniform convergence of $\mathsf{P}_t^{(n)} f$ towards f.

is measurable and for every $x \in M$,

$$\mathsf{Q}(d\varphi)\text{-a.s.,} \quad \mathcal{J}(\varphi)(x) = \varphi(x), \tag{1.3}$$

i.e. $\mathcal{J}$ is a measurable modification of the identity mapping on $(F, \mathcal{F}, \mathsf{Q})$. We call it a measurable presentation of Q.

Let Q_1 and Q_2 be two probability measures on $(F, \mathcal{F})$. Assume that Q_1 is regular. Let $\mathcal{J}$ be a measurable presentation of Q_1. Then the mapping

$$\begin{aligned}(F^2, \mathcal{F}^{\otimes 2}) &\to (F, \mathcal{F}),\\ (\varphi_1, \varphi_2) &\mapsto \mathcal{J}(\varphi_1) \circ \varphi_2\end{aligned}$$

is measurable. Moreover, if $\mathcal{J}'$ is another measurable presentation of Q_1, then for every $x \in M$,

$$\mathsf{Q}_1(d\varphi_1) \otimes \mathsf{Q}_2(d\varphi_2)\text{-a.s.,} \quad \mathcal{J}(\varphi_1) \circ \varphi_2(x) = \mathcal{J}'(\varphi_1) \circ \varphi_2(x). \tag{1.4}$$

Note that $(\varphi_1, \varphi_2) \mapsto \mathcal{J}(\varphi_1) \circ \varphi_2$ is measurable, but $(\varphi_1, \varphi_2) \mapsto \varphi_1 \circ \varphi_2$ is not measurable.

Definition 1.1.3. The *convolution product* of Q_1 and Q_2, denoted by $\mathsf{Q}_1 * \mathsf{Q}_2$, is the law of the random variable $(\varphi_1, \varphi_2) \mapsto \mathcal{J}(\varphi_1) \circ \varphi_2$ defined on the probability space $(F^2, \mathcal{F}^{\otimes 2}, \mathsf{Q}_1 \otimes \mathsf{Q}_2)$. A *convolution semigroup* on $(F, \mathcal{F})$ is a family $(\mathsf{Q}_t)_{t \geq 0}$ of regular probability measures on $(F, \mathcal{F})$ such that for all nonnegative s and t, $\mathsf{Q}_{s+t} = \mathsf{Q}_s * \mathsf{Q}_t$.

A convolution semigroup $(\mathsf{Q}_t)_{t \geq 0}$ on $(F, \mathcal{F})$ is called Feller if

(i) for all $f \in C(M)$, $\lim_{t \to 0} \sup_{x \in M} \int (f \circ \varphi(x) - f(x))^2 \mathsf{Q}_t(d\varphi) = 0$;

(ii) for all $f \in C(M)$ and $t \geq 0$, $\lim_{d(x,y) \to 0} \int (f \circ \varphi(x) - f \circ \varphi(y))^2 \mathsf{Q}_t(d\varphi) = 0$.

Let $(\mathsf{Q}_t)_{t \geq 0}$ be a Feller convolution semigroup on $(F, \mathcal{F})$. For all $n \geq 1$, $f \in C(M^n)$ and $x \in M^n$ set

$$\mathsf{P}_t^{(n)} f(x) = \int f \circ \varphi^{\otimes n}(x)\, \mathsf{Q}_t(d\varphi). \tag{1.5}$$

Then $(\mathsf{P}_t^{(n)},\ n \geq 1)$ is a compatible family of Feller semigroups on M satisfying

$$\mathsf{P}_t^{(2)} f^{\otimes 2}(x, x) = \mathsf{P}_t f^2(x) \tag{1.6}$$

for all $f \in C(M)$, $x \in M$ and $t \geq 0$. *The semigroup* $(\mathsf{Q}_t)_{t \geq 0}$ *is uniquely determined by* $(\mathsf{P}_t^{(n)},\ n \geq 1)$.

In the following we will consider only probability spaces $(\Omega, \mathcal{A}, \mathsf{P})$ which are separable, i.e., the corresponding Hilbert space $L^2(\Omega, \mathcal{A}, \mathsf{P})$ is separable.

Definition 1.1.4. Let $(\Omega, \mathcal{A}, \mathsf{P})$ be a probability space and $(T_h)_{h\in\mathbb{R}}$ a one parameter group of P-preserving L^2-continuous transformations of Ω. A family of $(F, \mathcal{F})$-valued random variables $(\varphi_{s,t},\ s \le t)$ is called a *measurable stochastic flow of mappings* if for all $s \le t$ the mapping

$$(M \times \Omega, \mathcal{B}(M) \otimes \mathcal{A}) \to (M, \mathcal{B}(M)),$$
$$(x, \omega) \mapsto \varphi_{s,t}(x, \omega)$$

is measurable and if it satisfies the following properties.

(a) (Cocycle property) For all $s < u < t$ and $x \in M$, P-almost surely, $\varphi_{s,t}(x) = \varphi_{u,t} \circ \varphi_{s,u}(x)$.

(b) (Stationarity) For all $s \le t$, $\varphi_{s+h,t+h} = \varphi_{s,t} \circ T_h$.

(c) The flow has independent increments, i.e. for all $t_1 < t_2 < \cdots < t_n$, the family $\{\varphi_{t_i,t_{i+1}},\ 1 \le i \le n-1\}$ is independent.

(d) For every $f \in C(M)$, $\lim_{(u,v)\to(s,t)} \sup_{x\in M} \mathsf{E}[(f\circ\varphi_{s,t}(x) - f\circ\varphi_{u,v}(x))^2] = 0$.

(e) For all $f \in C(M)$ and $s \le t$, $\lim_{d(x,y)\to 0} \mathsf{E}[(f \circ \varphi_{s,t}(x) - f \circ \varphi_{s,t}(y))^2] = 0$.

Let $\varphi = (\varphi_{s,t},\ s \le t)$ be a stochastic flow of mappings. For all $n \ge 1$, $f \in C(M^n)$ and $x \in M^n$ set

$$\mathsf{P}_t^{(n)} f(x) = \mathsf{E}[f \circ \varphi_{0,t}^{\otimes n}(x)]. \tag{1.7}$$

Then $(\mathsf{P}_t^{(n)},\ n \ge 1)$ is a compatible family of Feller semigroups on M satisfying (1.6). *The law of φ is uniquely determined by $(\mathsf{P}_t^{(n)},\ n \ge 1)$.*

Theorem 1.1.5. 1) *Let $(\mathsf{P}_t^{(n)},\ n \ge 1)$ be a compatible family of Feller semigroups on M satisfying*

$$\mathsf{P}_t^{(2)} f^{\otimes 2}(x, x) = \mathsf{P}_t f^2(x) \tag{1.8}$$

for all $f \in C(M)$, $x \in M$ and $t \ge 0$. Then there exists a unique Feller convolution semigroup $(\mathsf{Q}_t)_{t\ge 0}$ on $(F, \mathcal{F})$ such that for all $n \ge 1$, $t \ge 0$, $f \in C(M^n)$ and $x \in M^n$,

$$\mathsf{P}_t^{(n)} f(x) = \int f \circ \varphi^{\otimes n}(x)\, \mathsf{Q}_t(d\varphi). \tag{1.9}$$

2) *For every Feller convolution semigroup $\mathsf{Q} = (\mathsf{Q}_t)_{t\ge 0}$ on $(F, \mathcal{F})$ there exists a stochastic flow of mappings associated with Q (or equivalently with $(\mathsf{P}_t^{(n)},\ n \ge 1)$).*

Let $V, V_1, \ldots, V_k$ be bounded Lipschitz vector fields on a smooth locally compact manifold M. We also assume that $V_1, \ldots, V_k$ are C^1. Let $W^1, \ldots, W^k$ be k independent real white noises. We consider the following SDE on M:

$$dX_t = \sum_{i=1}^{k} V_i(X_t) \circ dW_t^i + V(X_t)\, dt, \quad t \in \mathbb{R}. \tag{1.10}$$

From the usual theory of strong solutions of SDEs (see for example [12]) it is possible to construct a stochastic flow of diffeomorphisms $(\varphi_{s,t},\ s \le t)$ such that for every $x \in M$, $\varphi_{s,t}(x)$ is a strong solution of the SDE (1.10) with $\varphi_{s,s}(x) = x$.

Using this stochastic flow, it is possible to construct a compatible family of Markovian semigroups $(\mathsf{P}_t^{(n)},\ n \ge 1)$ with

$$\mathsf{P}_t^{(n)} h(x_1, \dots, x_n) = \mathsf{E}[h(\varphi_{0,t}(x_1), \dots, \varphi_{0,t}(x_n))] \tag{1.11}$$

for $h \in C(M^n)$ and $x_1, \dots, x_n$ in M. It is easy to check that these semigroups are Feller and that the canonical stochastic flow of maps associated with this family of semigroups is equal in law to $(\varphi_{s,t},\ s \le t)$.

1.2. Flows of transition kernels. We denote by $\mathcal{P}(M)$ the space of probability measures on M, equipped with the weak convergence topology. $\mathcal{P}(M)$ is a compact metric space. Let us recall that a kernel K on M is a measurable mapping from M into $\mathcal{P}(M)$, M and $\mathcal{P}(M)$ being equipped with their Borel σ-fields. For all $f \in C(M)$ and $x \in M$, $Kf(x)$ denotes $\int f(y)\, K(x, dy)$. For every $\mu \in \mathcal{P}(M)$, μK denotes the probability measure defined by $\int f(y)\, \mu K(dy) = \int Kf(x)\, \mu(dx)$. We denote by E the space of all kernels on M, and we equip E with the σ-field $\mathcal{E}$ generated by the mappings $K \mapsto \mu K$ for every $\mu \in \mathcal{P}(M)$. Convolution semigroups on the space of kernels can be defined in a similar way as on the space of measurable maps (cf. [16]).

Definition 1.2.1. Let $(\Omega, \mathcal{A}, \mathsf{P})$ be a probability space and $(T_h)_{h\in\mathbb{R}}$ a one parameter group of P-preserving L^2-continuous transformations of Ω. Then a family of $(E, \mathcal{E})$-valued random variables $(K_{s,t},\ s \le t)$ is called a (*measurable*) *stochastic flow of kernels* if for all $s \le t$,

$$(x, \omega) \mapsto K_{s,t}(x, \omega) \tag{1.12}$$

is a measurable mapping from $(M \times \Omega, \mathcal{B}(M) \otimes \mathcal{A})$ onto $(\mathcal{P}(M), \mathcal{B}(\mathcal{P}(M)))$ and if it satisfies the following properties.

(a) (Cocycle property) For all $s < u < t$ and $x \in M$, P-almost surely, for every $f \in C(M)$, $K_{s,t}f(x) = K_{s,u}(K_{u,t}f)(x)$.

(b) (Stationarity) For all $s \le t$, $K_{s+h,t+h} = K_{s,t} \circ T_h$.

(c) The flow has independent increments, i.e. for all $t_1 < t_2 < \cdots < t_n$, the family $\{K_{t_i,t_{i+1}},\ 1 \le i \le n-1\}$ is independent.

(d) For every $f \in C(M)$,

$$\lim_{(u,v)\to(s,t)} \sup_{x\in M} \mathsf{E}[(K_{s,t}f(x) - K_{u,v}f(x))^2] = 0. \tag{1.13}$$

(e) For all $f \in C(M)$ and $s < t$,

$$\lim_{d(x,y)\to 0} \mathsf{E}[(K_{s,t}f(x) - K_{s,t}f(y))^2] = 0. \tag{1.14}$$

Let $(K_{s,t},\ s \leq t)$ be a stochastic flow of kernels. For all $n \geq 1,\ f \in C(M^n)$ and $x \in M^n$ set

$$\mathsf{P}_t^{(n)} f(x) = \mathsf{E}[K^{\otimes n} f(x)]. \tag{1.15}$$

Then $(\mathsf{P}_t^{(n)},\ n \geq 1)$ is a compatible family of Feller semigroups on M.

Theorem 1.2.2. 1) *For every compatible family* $(\mathsf{P}_t^{(n)},\ n \geq 1)$ *of Feller semigroups on M there exists a unique Feller convolution semigroup* $(\nu_t)_{t\geq 0}$ *on* $(E, \mathcal{E})$ *such that for all* $n \geq 1,\ t \geq 0,\ f \in C(M^n)$ *and* $x \in M^n$*,*

$$\mathsf{P}_t^{(n)} f(x) = \int K^{\otimes n} f(x)\ \nu_t(dK). \tag{1.16}$$

2) *For every Feller convolution semigroup* $\nu = (\nu_t)_{t\geq 0}$ *on* $(E, \mathcal{E})$ *there exists a stochastic flow of kernels associated with* ν *(or equivalently with* $(\mathsf{P}_t^{(n)},\ n \geq 1)$*).*

Remark 1.2.3. If (1.6) is satisfied the stochastic flow of kernels K is induced by a stochastic flow of mappings φ.

2. Noise and stochastic flows

The definition of a noise we give here is very close to the one given by Tsirelson in [25].

Definition 2.1.1. A *noise* consists of a separable probability space $(\Omega, \mathcal{A}, \mathsf{P})$, a one parameter group $(T_h)_{h\in\mathbb{R}}$ of P-preserving L^2-continuous transformations of Ω and a family $\{\mathcal{F}_{s,t},\ -\infty \leq s \leq t \leq \infty\}$ of sub-σ-fields of $\mathcal{A}$ such that

(a) T_h maps $\mathcal{F}_{s,t}$ onto $\mathcal{F}_{s+h,t+h}$ for all $h \in \mathbb{R}$ and $s \leq t$,

(b) $\mathcal{F}_{s,t}$ and $\mathcal{F}_{t,u}$ are independent for all $s \leq t \leq u$,

(c) $\mathcal{F}_{s,t} \vee \mathcal{F}_{t,u} = \mathcal{F}_{s,u}$ for all $s \leq t \leq u$.

A classical white noise or a stationary Poisson measure clearly define a noise in this sense.

A square integrable random variable with zero mean is said to belong to the first chaos if the sum of its conditional expectations with respect to the fields associated with disjoint intervals is the conditional expectation with respect to the field associated with the union of these intervals. The noise is called black when the first chaos reduces to zero. Clearly, the white noise or the Poisson noise are not black.

Let $(\Omega, \mathcal{A}, \mathsf{P}_\nu)$ denote the probability space of a stochastic flow of kernels $K = (K_{s,t},\ s \leq t)$ associated with a Feller convolution semigroup ν.

For all $-\infty \leq s \leq t \leq \infty$ let $\mathcal{F}_{s,t}$ be the sub-σ-field of $\mathcal{A}$ generated by the random variables $K_{u,v}$ for all $s \leq u \leq v \leq t$. Then the cocycle property of K implies that $N_\nu := (\Omega, \mathcal{A}, (\mathcal{F}_{s,t})_{s\leq t}, \mathsf{P}_\nu, (T_h)_{h\in\mathbb{R}})$ is a noise (T_h is L^2-continuous by the Feller property). We call it the noise generated by the flow K.

3. Stochastic coalescing flows

Starting from a compatible family of Feller semigroups, under the hypothesis that the two-point motion hits the diagonal almost surely, we construct another compatible family of Feller semigroups to which is associated a stochastic coalescing flow. It appears that the stochastic flow of kernels associated with the first family of semigroups can be recovered by filtering the stochastic coalescing flow with respect to a sub-noise of an extension of its noise.

Finally, we give the example of Arratia's flow ([2]), which describes a space-time continuum of independent Brownian motions sticking together when they meet. The construction of a stochastic coalescing flow solution of SDE's will be presented in the next sections (see also [11], [5]).

3.1. Definition and construction

Definition 3.1.1. A stochastic flow of mappings on M, $(\varphi_{s,t},\ s \le t)$, is called a *stochastic coalescing flow* if for all $(x, y) \in M^2$, $T_{x,y} = \inf\{t \ge 0,\ \varphi_{0,t}(x) = \varphi_{0,t}(y)\}$ is finite and for every $t \ge T_{x,y}$, $\varphi_{0,t}(x) = \varphi_{0,t}(y)$ almost surely.

This definition depends only on the two-point motion.

Let $(\mathsf{P}_t^{(n)},\ n \ge 1)$ be a compatible family of Feller semigroups on a compact separable metric space M, $\nu = (\nu_t)_{t\in\mathbb{R}}$ the associated Feller convolution semigroup on $(E, \mathcal{E})$ and K_t the associated flow of kernels. Let $\Delta_n = \{x \in M^n,$ there exists $i \ne j$ such that $x_i = x_j\}$ and $T_{\Delta_n} = \inf\{t \ge 0,\ X_t^{(n)} \in \Delta_n\}$, where $X_t^{(n)}$ denotes the n-point motion, i.e. the Markov process on M^n associated with the semigroup $\mathsf{P}_t^{(n)}$. Denoting by $\mathsf{P}_{(x,y)}^{(2)}$ the law of the Markov process associated with $\mathsf{P}_t^{(2)}$ starting from (x, y) and Δ_2 by Δ, assume that for all $t > 0$ and $x \in M$,

$$\lim_{y\to x} \mathsf{P}_{(x,y)}^{(2)}[\{T_\Delta > t\}] = 0,$$

and that for all x and y in M, $\mathsf{P}_{(x,y)}^{(2)}[T_\Delta < \infty] = 1$.

Theorem 3.1.2. *There exists a unique compatible family* $(\mathsf{P}_t^{(n),c},\ n \ge 1)$ *of Feller semigroups on M such that if $X^{(n),c}$ is the associated n-point motion and* $T_{\Delta_n}^c = \inf\{t \ge 0,\ X_t^{(n),c} \in \Delta_n\}$, *then*

- $(X_t^{(n),c},\ t \le T_{\Delta_n}^c)$ *is equal in law to* $(X_t^{(n)},\ t \le T_{\Delta_n})$,
- *for* $t \ge T_{\Delta_n}^c$, $X_t^{(n),c} \in \Delta_n$.

Moreover, $(\mathsf{P}_t^{(n),c},\ n \ge 1)$ *satisfies* (1.6) *and is associated with a coalescing flow* $\varphi_{s,t}^c$.

We denote by ν^c the associated Feller convolution semigroup. An important result is the following:

Theorem 3.1.3. *There is a joint realisation of K and φ^c such that* $K_{s,t}g(y) = \mathsf{E}[\varphi^c_{s,t}(y)|\sigma(K)]$

We say that the convolution semigroup ν^c weakly dominates ν.

3.2. Arratia's coalescing flow of independent Brownian motions. The first example of coalescing flows was given by Arratia [2]. On $\mathbb{R}$, or on the unit circle, let P_t be the semigroup of a Brownian motion. With this semigroup we define the compatible family $(\mathsf{P}_t^{\otimes n},\ n \geq 1)$ of Feller semigroups. Note that the n-point motion of this family of semigroups is given by n independent Brownian motions. Let us also remark that the canonical stochastic flow of kernels associated with this family of semigroups is not random and is given by $(\mathsf{P}_{t-s},\ s \leq t)$.

Let $(\mathsf{P}_t^{(n)},\ n \geq 1)$ be the compatible family of Markovian coalescent semigroups associated with $(\mathsf{P}_t^{\otimes n},\ n \geq 1)$. Note that the n-point motion of this family of semigroups is given by n independent Brownian motions which stick together when they meet.

Theorem 3.2.1. *The family* $(\mathsf{P}_t^{(n)},\ n \geq 1)$ *is constituted of Feller semigroups and is associated with a coalescing flow. The noise defined by this flow is black*

Blackness of the noise was first proved in [25] and then in a different way in [17]. It may seem a paradox but note that the increments of a one point motion between two times depend on the position of that point at the first time and not only on the increment of the flow of maps. In the latter paper a related family of flows of kernels, called sticky flows, was also constructed. The associated n point motions are given by Brownian paths which are independent except they stick together (during a Cantor type set of times of positive Lebesgue measure) when they meet. These flows interpolate between the heat flow and Arratia's flow. They also define a black noise. Any flow of kernels induces naturally a Markov process on measures which has often an invariant probability distribution. For sticky flows it is explicitly given in terms of the Poisson Dirichlet distribution. Finally, let us mention that a discrete model converging to these flows was presented in [13].

4. Tanaka's equation

Tanaka's stochastic differential equation (SDE) is one of the simplest examples of an SDE that does not have a strong solution in the usual sense. The objective is to apply to this example the theory of flows of transition kernels and to classify all the solutions of Tanaka's SDE, extended to transition kernels. It is shown that they can be characterized by a probability measure on [0, 1]. The domination and the weak domination relations (defined in [15]) between different solutions are then fully understood in terms of barycenter and balayage of the associated measures.

On a probability space $(\Omega, \mathcal{F}, \mathsf{P})$ let $W = (W_{s,t},\ s \le t)$ be a real white noise and $K = (K_{s,t},\ s \le t)$ (resp. $\varphi = (\varphi_{s,t},\ s \le t)$) be a stochastic flow of kernels (resp. flow of measurable maps) on the real line. Recall that for all $s \le t$, $K_{s,t} : \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is measurable, with $\mathcal{P}(\mathbb{R})$ denoting the set of probability measures on $\mathbb{R}$ equipped with the topology of weak convergence. We say that (K, W) solves Tanaka's SDE if for all $s \le t$, $f \in C^2_K(\mathbb{R})$ and $x \in \mathbb{R}$,

$$K_{s,t} f(x) = f(x) + \int_s^t K_{s,u}(f' \mathrm{sgn})(x) W(du) + \frac{1}{2} \int_s^t K_{s,u}(f'')du, \tag{4.1}$$

with $\mathrm{sgn}(x) = 1_{x \ge 0} - 1_{x<0}$. Note that (4.1) is a generalization of the SDE

$$dX_t = \mathrm{sgn}(X_t) dW_t,$$

where $W_t = W_{0,t} 1_{t \ge 0} - W_{t,0} 1_{t<0}$.

It can be shown that this implies that $\sigma(W) \subset \sigma(K)$. Let N^K be the noise of K. The noise N^W of W is a subnoise of N^K. So we can simply say that K solves Tanaka's SDE (since W is a function of K). We say that a flow of maps φ solves Tanaka's SDE if δ_φ solves Tanaka's SDE. The law of a solution K is given by a Feller convolution semigroup $\nu = (\nu_t,\ t \ge 0)$, where ν_t is the law of $K_{0,t}$.

Two particular solutions of Tanaka's SDE are given in [15]: the coalescing solution φ^c and the Wiener solution K^W. The solution K^W is the only solution of Tanaka's SDE such that $N^K = N^W$, and φ^c is the only flow of maps solution of Tanaka's SDE. The Wiener solution can be obtained by filtering the coalescing solution: $K^W = \mathsf{E}[\delta_\varphi | W]$. An explicit expression of K^W can be given. For $x \in \mathbb{R}$ set $\tau_x = \inf\{t > 0,\ W_{0,t} = -|x|\}$. Let $W^+ = (W_t^+,\ t \ge 0)$ be defined by

$$W_t^+ = W_{0,t} - \inf_{s \le t} W_{0,s}.$$

It is well known that the law of $(W_t^+)_{t \ge 0}$ and the law of $(|W_t|)_{t \ge 0}$ coincide. Note that $W_{0,\cdot}$ can be recovered out of W^+ by Doob–Meyer decomposition. Then for $t \ge 0$,

$$K_{0,t}^W(x) = \delta_{x + \mathrm{sgn}(x) W_{0,t}} 1_{\{t \le \tau_x\}} + \frac{1}{2}(\delta_{W_t^+} + \delta_{-W_t^+}) 1_{\{t > \tau_x\}}. \tag{4.2}$$

Let θ_h^W be the shift operator such that $W_{s,t} \circ \theta_h^W = W_{s+h,t+h}$. Then for all $s < t$, $K_{s,t}^W = K_{0,t-s}^W \circ \theta_s^W$. The coalescing solution φ^c can be defined by the consistent family of its n-point motions obtained by transforming the n-point motion associated with K^W into a coalescing motion. A more explicit definition can be given in this special case, as is shown in [18], where we also prove the following result:

Theorem 4.1.1. a) *Each solution K of Tanaka's SDE verifies $K^W = \mathsf{E}[K|W]$ (this means in particular that the support of K has at most two points). It defines a probability measure m on $[0, 1]$ with mean $1/2$, which is the law of $\int_0^\infty K_{0,t}(0, dy)$ for all $t > 0$.*

b) *The mapping defined in* a) *is a bijection between solutions of* (4.1) *and probability measures on* [0, 1] *with mean* 1/2*. The Feller convolution semigroup associated with a measure m is denoted* $\{\nu_t^m,\ t \geq 0\}$ *or* ν^m.

c) K^W *is associated with* $\delta_{1/2}$ *and* φ^c *with* $\frac{1}{2}(\delta_0 + \delta_1)$.

Let us now describe the domination relations.

Definition 4.1.2. Let m_1 and m_2 be probability measures on [0, 1].

a) m_1 is swept by m_2 if and only if for all positive convex function f, $\int f dm_2 \leq \int f dm_1$.

b) m_2 is a barycenter of m_1 if and only if there exists a measurable map $\psi : [0,1] \to [0,1]$ such that $\psi^* m_1 = m_2$ and $\psi^*(I \cdot m_1) = I \cdot m_2$ (where I denotes the identity function).

It can easily be seen that a) and b) define partial order relations. The order defined in a) is the balayage order. The fact that m_2 is a barycenter of m_1 is equivalent to saying that if U_1 is a random variable of law m_1, then there exists a $\sigma(U_1)$-measurable random variable U_2 of law m_2 such that $\mathsf{E}[U_1|U_2] = U_2$.

In [16], a domination and a weak domination relation between (laws of) stochastic flow of kernels is defined: Let ν^1 and ν^2 be two Feller convolution semigroups. We recall that Definition 3.3 in [16] essentially says that ν^1 dominates ν^2 if and only if there is a joint realisation (K^1, K^2) such that K^1 (resp. K^2) is a stochastic flow of kernels associated to ν^1 (resp. to ν^2) satisfying $\mathsf{E}[K^1|K^2] = K^2$ and $\sigma(K^2) \subset \sigma(K^1)$. One says that ν^1 weakly dominates ν^2 when only the conditional expectation assumption is verified ($\sigma(K^2)$ needs not be a sub-σ-field of $\sigma(K^1)$). A full understanding of the solutions of a general SDE should involve a classification of the solutions according to these domination relations. As we will see in the following section, this is not achieved yet even in relatively simple cases.

Theorem 4.1.3. *Let* m_1 *and* m_2 *be two probability measures on* [0, 1] *with mean* 1/2*.*

a) ν^{m_1} *dominates* ν^{m_2} *if and only if* m_2 *is a barycenter of* m_1.

b) ν^{m_1} *weakly dominates* ν^{m_2} *if and only if* m_1 *is swept by* m_2.

5. Stochastic flows of kernels and SDEs: an example on the circle

Notation. In all the following we will denote by $\mathbb{S}$ the unit circle $\mathbb{R}/2\pi\mathbb{Z}$, by m the Lebesgue measure on $\mathbb{S}$ and by $\mathcal{P}(\mathbb{S})$ the set of Borel probability measures on $\mathbb{S}$.

Let $(\mathcal{W}, \mathcal{F}^W, \mathsf{P}_W)$ be the canonical probability space of a sequence of independent Wiener processes $(W_t^k,\ k \geq 0,\ t \geq 0)$. For all $s < t$ let $\mathcal{F}_{s,t}^W$ denote the σ-field generated by the random variables $W_v^k - W_u^k$, $s \leq u < v \leq t$ and $k \geq 0$. Being given $(a_k)_{k\geq 0}$ a sequence of nonnegative numbers such that $\sum_{k\geq 0} a_k^2 < \infty$, we set $C(z) = \sum_{k\geq 0} a_k^2 \cos(kz)$. Note that all real positive definite functions on $\mathbb{S}$ can be written in this form and that $C(0) = \sum_{k\geq 0} a_k^2$.

5.1. Flows of diffeomorphisms. Assume that $\sum_{k\geq 1} k^2 a_k^2 < \infty$. Then by a stochastic version of Gronwall's lemma it can be shown that for each $x_0 \in \mathbb{S}$ the stochastic differential equation (SDE)

$$\begin{aligned} x_t &= x_0 + a_0 W_t^0 \\ &\quad + \sum_{k\geq 1} a_k \left(\int_0^t \sin(kx_s) dW_s^{2k-1} + \int_0^t \cos(kx_s) dW_s^{2k} \right) \end{aligned} \tag{5.1}$$

has a unique strong solution. These solutions can be considered jointly to form a stochastic flow of diffeomorphisms $(\varphi_{s,t})_{s<t}$. Set $\varphi_t = \varphi_{0,t}$.

Note that the one point motion $x_t := \varphi_t(x)$ is a Brownian motion on $\mathbb{S}$ starting at x. Denote the associated heat semigroup P_t. For $h \in C^2(\mathbb{S}^2)$ set $A^{(2)}h(x, y) = \frac{C(0)}{2}(\partial_{xx}^2 h(x, y) + \partial_{yy}^2 h(x, y)) + C(x-y)\partial_{xy}^2 h(x, y)$. The two point motion $(x_t, y_t) := (\varphi_t(x), \varphi_t(y))$ is a diffusion on $\mathbb{S}^2$ satisfying

- for all $h \in C^2(\mathbb{S}^2)$, $h(x_t, y_t) - \int_0^t A^{(2)}h(x_s, y_s)ds$ is an L^2-martingale.

This in particular implies the following:

- For all $x, y \in \mathbb{S}$, $f \in C(\mathbb{S})$, if $Z_t = \varphi_t(x) - \varphi_t(y)$, Z_t is a diffusion on $\mathbb{S}$ and $f(Z_t) - \int_0^t (C(0) - C(Z_s)) f''(Z_s)ds$ is an L^2-martingale.

For all $x, y \in \mathbb{S}$, $\lim_{t\to\infty}(\varphi_t(x) - \varphi_t(y)) = 0$. Using the isotropy one can compute the Lyapounov exponent of the flow: for all $x \in \mathbb{S}$,

$$\lim_{t\to\infty} \frac{1}{t} \log |\varphi_t'(x)| = -\frac{1}{2} \sum_{k\geq 1} k^2 a_k^2.$$

The Lyapounov exponent of the flow being negative, the flow is stable. In the particular case $a_k^2 = k^{-(1+\alpha)}$, for $\alpha > 2$ the condition $\sum_{k\geq 1} k^2 a_k^2 < \infty$ is satisfied. When α is close to the boundary case, $\alpha = 2$, the Lyapounov exponent gets close to $-\infty$. In the following we will define stochastic flows corresponding to the case $\alpha \leq 2$ which are not flows of diffeomorphisms.

5.2. Wiener chaos expansion: Lipschitz case. Suppose that $\sum_k k^2 a_k^2 < \infty$ and let $\varphi_{s,t}$ be the flow defined in the previous section. For any function $f \in C(\mathbb{S})$, $x \in \mathbb{S}$ and $s \leq t$, $f \circ \varphi_{s,t}(x)$ belongs to the Wiener space $L^2(\mathsf{P}_W)$. Following the original idea of [26], its Wiener chaos expansion can be explicitly computed as follows.

Proposition 5.2.1. *For all $s \le t$ and $f \in C(\mathbb{S})$,*

$$f \circ \varphi_{s,t}(x) = \mathsf{P}_{t-s} f(x) + \sum_{n \ge 1} J^n_{s,t} f(x) \quad \text{in } L^2(\mathsf{P}_W), \tag{5.2}$$

where J^n is defined recursively as follows (denoting c_k the function $x \mapsto \cos(kx)$ and s_k the function $x \mapsto \sin(kx)$):

$$\begin{aligned} J^{n+1}_{s,t} f(x) = a_0 \int_s^t & J^n_{s,u}((\mathsf{P}_{t-u} f)')(x) dW^0_u \\ &+ \sum_{k \ge 1} a_k \Bigg(\int_s^t J^n_{s,u}(s_k(\mathsf{P}_{t-u} f)')(x) dW^{2k-1}_u \\ &\qquad + \int_s^t J^n_{s,u}(c_k(\mathsf{P}_{t-u} f)')(x) dW^{2k}_u \Bigg) \end{aligned}$$

for $n \ge 0$ with $J^0_{s,t} = \mathsf{P}_{t-s}$.

Remark 5.2.2. The chaos expansion (5.2) can be extended to all $f \in L^2(m)$, the two terms being equal in $L^2(m \otimes \mathsf{P}_W)$.

5.3. Non-Lipschitz case. From now on we assume $\sum_{k \in \mathbb{N}} k^2 a_k^2 = \infty$. In this case, using Gronwall's inequality, the existence of a strong solution to the SDE (5.1) cannot be proven. But the series giving the Wiener chaos expansion of $f \circ \varphi_{s,t}$ in the Lipschitz case for $f \in L^2(m)$ also converges in $L^2(m \otimes \mathsf{P}_W)$ in the non-Lipschitz case.

We can construct a family $S^n_{s,t}$ of random operators acting of $L^2(m)$ recursively: let $S^0_{s,t} = \mathsf{P}_{t-s}$ and for $f \in L^2(m)$ and $n \ge 0$ set

$$\begin{aligned} S^{n+1}_{s,t} f = \mathsf{P}_{t-s} f + a_0 \int_0^t & S^n_{s,u}((\mathsf{P}_{t-u} f)') dW^0_u \\ &+ \sum_{k \ge 1} a_k \left(\int_s^t S^n_{s,u}(s_k(\mathsf{P}_{t-u} f)') dW^{2k-1}_u + \int_s^t S^n_{s,u}(c_k(\mathsf{P}_{t-u} f)') dW^{2k}_u \right). \end{aligned}$$

It can be seen that for all n, $\mathsf{E}[(S^n_{s,t} f)^2] \le \mathsf{P}_{t-s} f^2$ and

$$S^n_{s,t} f = \sum_{k=0}^{n} J^k_{s,t} f, \tag{5.3}$$

where $J^k_{s,t} f$ belongs to the k-th Wiener chaos. Thus all these terms are orthogonal and $S^n_{s,t} f$ converges in $L^2(m \otimes \mathsf{P}_W)$ towards a limit we denote by $S_{s,t} f$. The family $S = (S_{s,t})$ of random operators acting on $L^2(m)$ satisfies the following.

(i) Cocycle property: $S_{s,u} = S_{s,t} S_{t,u}$ for all $s < t < u$.

(ii) Stationary increments: for all $s \le t$, $S_{s,t}$ and $S_{0,t-s}$ have the same law.

(iii) Independent increments: for $t_0 \le \cdots \le t_n$, $S_{t_0,t_1}, \ldots, S_{t_{n-1},t_n}$ are independent.

(iv) Solution of the SDE

$$\begin{aligned} S_{s,t} f = f &+ a_0 \int_s^t S_{s,u}(f') dW_u^0 \\ &+ \sum_{k\ge1} a_k \left(\int_s^t S_{s,u}(s_k f') dW_u^{2k-1} + \int_s^t S_{s,u}(c_k f') dW_u^{2k} \right) \\ &+ \frac{C(0)}{2} \int_s^t S_{s,u} f'' du, \end{aligned} \tag{5.4}$$

for all $f \in H^2(\mathbb{S})$ and all $s < t$.

Moreover, S is the unique family of random operators acting on $L^2(m)$ verifying $\mathsf{E}[(S_{s,t} f)^2] \le \mathsf{P}_{t-s} f^2$, satisfying (i), (ii), (iii), (iv) and such that $S_{s,t}$ is $\mathcal{F}_{s,t}^W$-measurable.

Obviously $S_{s,t} 1 = 1$, and it can be proved that $S_{s,t}$ is nonnegative as follows. Consider an independent stationary Brownian motion B_t with diffusion coefficient $C(0)$ on $\mathbb{S}$. Set, for $k \ge 1$,

$$\begin{aligned} \widetilde{W}_t^{2k-1} = W_t^{2k-1} &+ a_k \int_0^t s_k(B_s) dB_s - a_k a_0 \int_0^t s_k(B_s) dW_s^0 \\ &- a_k \sum_{l\ge1} a_l \left(\int_0^t s_k s_l(B_s) dW_s^{2l-1} + \int_0^t s_k c_l(B_s) dW_s^{2l} \right) \end{aligned}$$

and, for $k \ge 0$,

$$\begin{aligned} \widetilde{W}_t^{2k} = W_t^{2k} &+ a_k \int_0^t c_k(B_s) dB_s - a_k a_0 \int_0^t c_k(B_s) dW_s^0 \\ &- a_k \sum_{l\ge1} a_l \left(\int_0^t c_k s_l(B_s) dW_s^{2l-1} + \int_0^t c_k c_l(B_s) dW_s^{2l} \right). \end{aligned}$$

These formulas are obtained by conditioning the "velocity differential" at time t and site B_t to be dB_t. Then $\widetilde{W}$ forms a family of independent Wiener processes. Set

$$\tilde{S}_{s,t} f(x) = \mathsf{E}[f(B_t) | \widetilde{W}, B_s = x].$$

It is clear that $\tilde{S}$ is nonnegative and that $\tilde{S}$ verifies the properties listed above ((i), (ii), (iii) and (iv)) with respect to $\widetilde{W}$. This implies $\tilde{S} = S$ and proves that S is nonnegative.

Two cases may occur:

(a) $S_{s,t} f^2 = (S_{s,t} f)^2$ for all $f \in L^\infty(m)$.

(b) $S_{s,t} f^2 > (S_{s,t} f)^2$ for some $f \in L^\infty(m)$, and in fact for all non constant $f \in L^\infty(m)$.

5.4. ***n*-point motions.** Let $\mathsf{P}_t^{(n)}$ be the family of random operators acting on $L^\infty(m^{\otimes n})$ defined by

$$\mathsf{P}_t^{(n)} f_1 \otimes \cdots \otimes f_n = \mathsf{E}[S_{0,t} f_1 \otimes \cdots \otimes S_{0,t} f_n].$$

Properties (i), (ii) and (iii) imply that $\mathsf{P}_t^{(n)}$ is a Markovian semigroup. As in the case of $\mathbb{R}^d$ or $\mathbb{S}^d$ studied in [16], one can show that the isotropy implies that $\mathsf{P}_t^{(n)}$ is a Feller semigroup acting on $C(\mathbb{S}^n)$.

The *n-point motion* of $(S_{s,t})$ is the diffusion on $\mathbb{S}^n$ associated with $\mathsf{P}_t^{(n)}$. The generator $A^{(n)}$ of this diffusion is given by

$$A^{(n)} = \frac{1}{2} \sum_{1 \le i,j \le n} C(x_i - x_j) \partial_{x_i} \partial_{x_j}. \tag{5.5}$$

The case (a) appears when the diagonal is absorbing for the two-point motion. If this is not the case we are in case (b).

5.5. Diffusive or coalescing? In case (a) it can be shown (using the Feller property) that there exists a flow of random mappings $\varphi = (\varphi_{s,t})$ such that for all $s \le t$ and all $f \in L^2(m)$, we have $S_{s,t} f = f \circ \varphi_{s,t}$ in $L^2(m \otimes \mathsf{P}_W)$. Furthermore, $\varphi_{s,t} \colon (\mathbb{S} \times \mathcal{W}, \mathcal{B}(\mathbb{S}) \otimes \mathcal{F}^W) \to (\mathbb{S}, \mathcal{B}(\mathbb{S}))$ is measurable and solves the SDE (5.1).

In case (b) it can be shown that there exists a flow of random kernels $K = (K_{s,t}^W)$ such that $S_{s,t} f = K_{s,t}^W f$ in $L^2(m \otimes \mathsf{P}_W)$ for all $s \le t$ and all $f \in L^2(m)$. The stochastic flow of kernels will be called *diffusive* when the kernels are not induced by maps, which clearly happens in case (b). This flow solves the SDE in the sense that for all $f \in C^2(\mathbb{S})$, $s \le t$ and $x \in \mathbb{S}$,

$$\begin{aligned} K_{s,t}^W f = f + \sum_{k \ge 1} a_k \left(\int_s^t K_{s,u}^W (s_k f') dW_u^{2k-1} + \int_s^t K_{s,u}^W (c_k f') dW_u^{2k} \right) \\ + a_0 \int_s^t K_{s,u}^W (f') dW_u^0 \quad + \quad \frac{C(0)}{2} \int_s^t K_{s,u}^W f'' du. \end{aligned} \tag{5.6}$$

In the following the flow φ (in case (a)) or the flow K^W (in case (b)) will be called the *Wiener solution* of the SDE (5.1). Since $(S_{s,t})$ is the unique solution of (5.4) which is $\mathcal{F}_{s,t}^W$-measurable, the Wiener solution φ (or K^W) is the unique solution of SDE (5.1) (or of (5.6)) which is $\mathcal{F}_{s,t}^W$-measurable.

A diffusive flow is called *diffusive with hitting* if the two-point motion hits the diagonal $\Delta = \{(x,x),\ x \in \mathbb{S}\}$.

The diffusion $z_t \in [0, 2\pi)$ such that $z_t = X_t - Y_t$ modulo 2π, where (X_t, Y_t) is the two point motion, has a natural scale. The speed measure m of this diffusion is given by $m(dz) = (C(0) - C(z))^{-1} dz$. Let κ be defined by $\kappa(z) = \int_\pi^z \frac{z-x}{C(0)-C(x)} dx$. Note that $\kappa(0+) = \infty$ implies that $m((0, 2\pi)) = \infty$.

Theorem 5.5.1. 1) *If* $\kappa(0^+) = \infty$ *then the Wiener solution is a stochastic flow of maps, which is not a coalescing flow.*

2) *If* $m((0, 2\pi)) = \infty$ *and* $\kappa(0^+) < \infty$ *then the Wiener solution is a coalescing flow.*

3) *If* $m((0, 2\pi)) < \infty$ *then the Wiener solution is a diffusive flow with hitting.*

Corollary 5.5.2. *Let* $a_k^2 = k^{-(1+\alpha)}$ *with* $\alpha > 0$.

1) *If* $\alpha > 2$, *then the Wiener solution is a stochastic flow of* C^1*-diffeomorphisms.*
2) *If* $\alpha = 2$ *then the Wiener solution is a stochastic flow of maps, which is not a coalescing flow.*
3) *If* $\alpha \in [1, 2)$ *then the Wiener solution is a coalescing flow.*
4) *If* $\alpha \in (0, 1)$ *then the Wiener solution is a diffusive flow with hitting.*

Remark 5.5.3. The case $\alpha = 2$ has been studied in [1], [8], [22]. It is shown in particular that the maps of the flow are homeomorphisms.

5.6. Extension of the noise and weak solution. Let $(\Omega, \mathcal{A}, \mathsf{P})$ be an extension of the probability space $(\mathcal{W}, \mathcal{F}^W, \mathsf{P}_W)$. We say that a measurable flow of maps $\varphi = (\varphi_{s,t})$ is a *weak solution* of (5.1) if it satisfies (5.1) without being $\mathcal{F}_{s,t}^W$-measurable. Similarly, a measurable flow of kernels $K = (K_{s,t})$ will be called *weak* (*generalized*) *solution* of the SDE (5.1) if it satisfies (5.6) without being $\mathcal{F}_{s,t}^W$-measurable.

We have seen that uniqueness is verified if one assumes in addition Wiener measurability: $K_{s,t}$ is $\mathcal{F}_{s,t}^W$-measurable for all $s \le t$.

In case (b) a different consistent system of Feller semigroups $\mathsf{P}_t^{(n),c}$ can be constructed by considering the coalescing n-point motion $X_t^{(n),c}$ associated with $X_t^{(n)}$, the n-point motion of the Wiener solution. A measurable flow of coalescing maps $\varphi_{s,t}^c$ whose n-point motion is $X_t^{(n),c}$ can be defined on an extension $(\Omega, \mathcal{A}, \mathsf{P})$ of the probability space $(\mathcal{W}, \mathcal{F}^W, \mathsf{P}_W)$. This coalescing flow also solves the SDE (5.1). It is a weak solution.

For $s \le t$ set $\mathcal{F}_{s,t}^c = \sigma(\varphi_{u,v}^c,\ s \le u \le v \le t)$. Then $(\mathcal{F}_{s,t}^c)_{s \le t}$ defines a noise. It can be seen (for details see [16]) that $\mathcal{F}_{s,t}^W \subset \mathcal{F}_{s,t}^c$ (this property also holds for any flow solution of SDE). This solution being different from the Wiener solution implies $\mathcal{F}_{s,t}^W \neq \mathcal{F}_{s,t}^c$. The noise $(\mathcal{F}_{s,t}^c)$ cannot be generated by Brownian motions. It is a non-classical noise (see also [27], [28]). The Wiener solution K^W can be recovered by filtering:

$$K_{s,t}^W f = \mathsf{E}[f \circ \varphi_{s,t}^c | \mathcal{F}_{s,t}^W], \quad \text{for all } f \in C(\mathbb{S}).$$

It can be shown that in case (a) there is no weak solution different in law from the Wiener solution. In case (b), $\varphi_{s,t}^c$ is the only solution which is a flow of maps. There are certainly other "intermediate" kernel solutions similar to the sticky flows, but they have not been constructed yet.

Final remarks. Similar results hold in a more general context, especially in the case of $\mathbb{S}^d$ and $\mathbb{R}^d$ (including $d = 1$). In fact, for isotropic flows in dimension $d \geq 2$, a different phase appears, in which the Wiener solution is a diffusive flow without hitting. This solution cannot be represented by filtering a coalescing solution defined on an extended probability space and there are no weak (generalized) solutions. In dimension 2 and 3 the coalescing phase (where the Wiener solution is a coalescing flow) and the phase of non uniqueness (where the Wiener solution is diffusive with hitting) still occurs.

Many important questions remain open: for example, the nature of the noises when they are not classical, the possible relations with rough paths ([20]), and the classifications of all solutions, starting with the isotropic case.

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