Estimation in inverse problems and second-generation wavelets

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Abstract. We consider the problem of recovering a function f when we receive a blurred (by a linear operator) and noisy version: $Y_\varepsilon = Kf + \varepsilon \dot{W}$. We will have as guides 2 famous examples of such inverse problems: the deconvolution and the Wicksell problem. The direct problem (K is the identity) isolates the denoising operation. It cannot be solved unless accepting to estimate a smoothed version of f: for instance, if f has an expansion on a basis, this smoothing might correspond to stopping the expansion at some stage m. Then a crucial problem lies in finding an equilibrium for m, considering the fact that for m large, the difference between f and its smoothed version is small, whereas the random effect introduces an error which is increasing with m. In the true inverse problem, in addition to denoising, we have to 'inverse the operator' K, an operation which not only creates the usual difficulties, but also introduces the necessity to control the additional instability due to the inversion of the random noise. Our purpose here is to emphasize the fact that in such a problem there generally exists a basis which is fully adapted to the problem, where for instance the inversion remains very stable: this is the singular value decomposition basis. On the other hand, the SVD basis might be difficult to determine and to numerically manipulate. It also might not be appropriate for the accurate description of the solution with a small number of parameters. Moreover, in many practical situations the signal provides inhomogeneous regularity, and its local features are especially interesting to recover. In such cases, other bases (in particular, localised bases such as wavelet bases) may be much more appropriate to give a good representation of the object at hand. Our approach here will be to produce estimation procedures keeping the advantages of a localisation properly without loosing the stability and computability of SVD decompositions. We will especially consider two cases. In the first one (which is the case of the deconvolution example) we show that a fairly simple algorithm (WAVE-VD), using an appropriate thresholding technique performed on a standard wavelet system, enables us to estimate the object with rates which are almost optimal up to logarithmic factors for any $\mathbb{L}_p$ loss function and on the whole range of Besov spaces. In the second case (which is the case of the Wicksell example where the SVD basis lies in the range of Jacobi polynomials) we prove that a similar algorithm (NEED-VD) can be performed provided one replaces the standard wavelet system by a second generation wavelet-type basis: the needlets. We use here the construction (essentially following the work of Petrushev and co-authors) of a localised frame linked with a prescribed basis (here Jacobi polynomials) using a Littlewood–Paley decomposition combined with a cubature formula. Section 5 describes the direct case ($K = I$). It has its own interest and will act as a guide for understanding the 'true' inverse models for a reader who is not familiar with nonparametric statistical estimation. It can be read first. Section 1 introduces the general inverse problem and describes the examples of deconvolution and Wicksell's problem. A review of standard methods is given with a special focus on SVD methods. Section 2 describes the WAVE-VD procedure. Section 3 and 4 give a description of the needlets constructions and the performances of the NEED-VD procedure.

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1. Inverse models

Let $\mathbb{H}$ and $\mathbb{K}$ be two Hilbert spaces. K is a linear operator: $f \in \mathbb{H} \mapsto Kf \in \mathbb{K}$. The standard linear ill-posed inverse problem consists of recovering a good approximation f_ε of f, solution of

$$g = Kf, \tag{1}$$

when only a perturbation g_ε of g is observed. In this paper we will consider the case where this perturbation is an additive stochastic white noise. Namely, we observe Y_ε defined by the following equation:

$$Y_\varepsilon = Kf + \varepsilon \dot{W}, \ \mathbb{H}, \ \mathbb{K}, \tag{2}$$

where ε is the amplitude of the noise. It is supposed to be a small parameter which will tend to 0. Our error will be measured in terms of this small parameter.

$\dot{W}$ is a $\mathbb{K}$-white noise: i.e. for any g, h in $\mathbb{K}$, $\xi(g) := (\dot{W}, g)_{\mathbb{K}}$, $\xi(h) := (\dot{W}, h)_{\mathbb{K}}$ form a random gaussian vector, centered, with marginal variance $\|g\|^2_{\mathbb{K}}$, $\|h\|^2_{\mathbb{K}}$, and covariance $(g, h)_{\mathbb{K}}$ (with the obvious extension when one considers k functions instead of 2).

Equation (2) means that for any g in $\mathbb{K}$, we observe $Y_\varepsilon(g) := (Y_\varepsilon, g)_{\mathbb{K}} = (Kf, g)_{\mathbb{K}} + \varepsilon\xi(g)$ where $\xi(g) \sim N(0, \|g\|^2)$, and $Y_\varepsilon(g)$, $Y_\varepsilon(h)$ are independent random variables for orthogonal functions g and h.

The case where K is the identity is called the 'direct model' and is summarized as a memento in Section 5. The reader who is unfamiliar with nonparametric statistical estimation is invited to consult this section, which will act as a guide for understanding the more general inverse models. In particular it is recalled therein that the model (2) is in fact an approximation of models appearing in real practical situations, for instance the case where (2) is replaced by a discretisation.

1.1. Two examples: the problem of deconvolution and Wicksell's problem

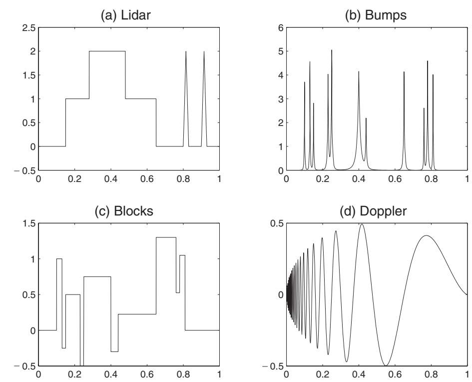
1.1.1. Deconvolution. The following problem is probably one of the most famous among inverse problems in signal processing. In the deconvolution problem we consider the following operator. In this case let $\mathbb{H} = \mathbb{K}$ be the set of square integrable periodic functions with the standard $\mathbb{L}_2([0, 1])$ norm and consider

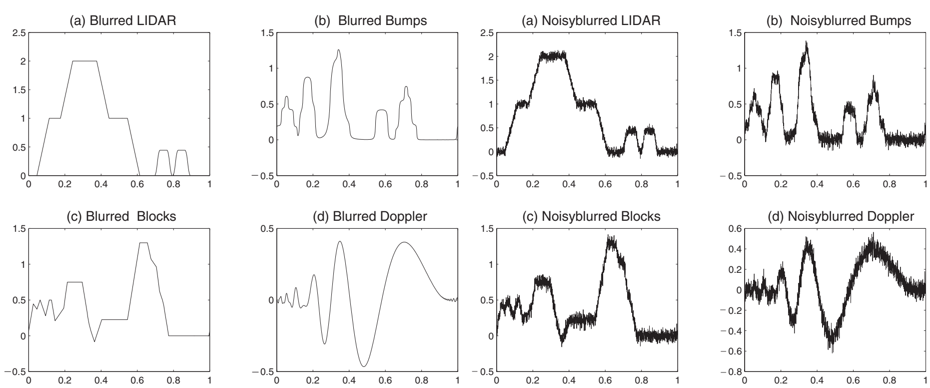
$$f \in \mathbb{H} \mapsto Kf = \int_0^1 \gamma(u - t) f(t)\, dt \in \mathbb{H}, \tag{3}$$

where γ is a known function of $\mathbb{H}$, which is generally assumed to be a regular function (often in the sense that its Fourier coefficients $\hat{\gamma}_k$ behave like $k^{-\nu}$). A very common example is also the box-car function: $\gamma(t) = \frac{1}{2a}\mathbb{I}\{[-a, a]\}(k)$.

The following figures show first four original signals to recover, which are well-known test-signals of the statistical literature. They provide typical features which are difficult to restore: bumps, blocks and Doppler effects. The second and third series of pictures show their deformation after blurring (i.e. convolution with a regular function) and addition of a noise. These figures show how the convolution regularizes the signal, making it very difficult to recover, especially the high frequency features. A statistical investigation of these signals can be found in [22].

A variant of this problem consists in observing $Y_1, \dots, Y_n$, n independent and identically distributed random variables where each Y_i may be written as $Y_i = X_i + U_i$, where X_i and U_i again are independent, the distribution of U_i is known and of density γ and we want to recover the common density of the X_i's. The direct problem is the case where $U_i = 0$, for all i, and is corresponding to a standard density estimation problem (see Section 5.1) . Hence the variables U_i are acting as perturbations of the X_i's, whose density is to be recovered.





1.1.2. Wicksell's problem. Another typical example is the following classical Wicksell problem [42]. Suppose a population of spheres is embedded in a medium. The spheres have radii that may be assumed to be drawn independently from a density f. A random plane slice is taken through the medium and those spheres that are intersected by the plane furnish circles the radii of which are the points of observation $Y_1, \dots, Y_n$. The unfolding problem is then to infer the density of the sphere radii from the observed circle radii. This unfolding problem also arises in medicine, where the spheres might be tumors in an animal's liver [36], as well as in numerous other contexts (biological, engineering,…), see for instance [9].

Following [42] and [23], Wicksell's problem corresponds to the following operator:

$$\mathbb{H} = \mathbb{L}_2([0,1], d\mu)\ d\mu(x) = (4x)^{-1}dx,$$
$$\mathbb{K} = \mathbb{L}_2([0,1], d\lambda)\ d\lambda(x) = 4\pi^{-1}(1-y^2)^{1/2}dy$$
$$Kf(y) = \frac{\pi}{4}y(1-y^2)^{-1/2}\int_y^1 (x^2-y^2)^{-1/2} f(x)d\mu.$$

Notice, however, that in this presentation, again in order to avoid additional technicalities, we handle this problem in the white noise framework, which is simpler than the original problem expressed above in density terms.

1.2. Singular value decomposition and projection methods. Let us begin with a quick description of well-known methods in inverse problems with random noise.

Under the assumption that K is compact, there exist 2 orthonormal bases (SVD bases) (e_k) of $\mathbb{H}$ and (g_k) of $\mathbb{K}$, respectively, and a sequence (b_k), tending to 0 when k goes to infinity, such that

$$Ke_k = b_k g_k, \quad K^* g_k = b_k e_k$$

if K^* is the adjoint operator.

For the sake of simplicity we suppose in the sequel that K and K^* are into. Otherwise we have to take care of the kernels of these operators. The bases (e_k) and (g_k) are called singular value bases, whereas the b_k's are simply called singular values.

Deconvolution. In this standard case simple calculations prove that the SVD bases (e_k) and (g_k) both coincide with the Fourier basis. The singular values are corresponding to the Fourier coefficients of the function γ:

$$b_k = \hat{\gamma}_k. \tag{4}$$

Wicksell. In this case, following [23], we have the following SVD:

$$e_k(x) = 4(k+1)^{1/2}x^2 P_k^{0,1}(2x^2-1),$$
$$g_k(y) = U_{2k+1}(y).$$

$P_k^{0,1}$ is the Jacobi polynomial of type (0, 1) with degree k, and U_k is the second type Chebyshev polynomial with degree k. The singular values are

$$b_k = \frac{\pi}{16}(1+k)^{-1/2}. \tag{5}$$

1.2.1. SVD method. The singular value decomposition (SVD) of K,

$$Kf = \sum_k b_k \langle f, e_k \rangle g_k,$$

gives rise to approximations of the type

$$f_\varepsilon = \sum_{k=0}^{N} b_k^{-1} \langle y_\varepsilon, g_k \rangle e_k,$$

where $N = N(\varepsilon)$ has to be chosen properly. This SVD method is very attractive theoretically and can be shown to be asymptotically optimal in many situations (see Mathé and Pereverzev [31], Cavalier and Tsybakov [6], Mair and Ruymgaart [29]). It also has the big advantage of performing a quick and stable inversion of the operator. However, it suffers from different types of limitations. The SVD bases might be difficult to determine as well as to numerically manipulate. Secondly, while these bases are fully adapted to describe the operator K, they might not be appropriate for the accurate description of the solution with a small number of parameters.

Also in many practical situations the signal provides inhomogeneous regularity, and its local features are especially interesting to recover. In such cases other bases (in particular localised bases such as wavelet bases) may be much more appropriate to give a good representation of the object at hand.

1.2.2. Projection methods. Projection methods which are defined as solutions of (1) restricted to finite dimensional subspaces $\mathbb{H}_N$ and $\mathbb{K}_N$ (of dimension N) also give rise to attractive approximations of f, by properly choosing the subspaces and the tuning parameter N (Dicken and Maass [10], Mathé and Pereverzev [31] together with their non linear counterparts Cavalier and Tsybakov [6], Cavalier et al. [7], Tsybakov [41], Goldenshluger and Pereverzev [19], Efromovich and Koltchinskii [16]). In the case where $\mathbb{H} = \mathbb{K}$ and K is a self-adjoint operator, the system is particularly simple to solve since the restricted operator K_N is symmetric positive definite. This is the so-called Galerkin method. Obviously, restricting to finite subspaces has similar effects and can also be seen as a *Tychonov regularisation*, i.e. minimizing the least square functional penalised by a regularisation term.

The advantage of the Galerkin method is to allow the choice of the basis. However the Galerkin method suffers from the drawback of being unstable in many cases.

Comparing the SVD and Galerkin methods exactly states one main difficulty of the problem. The possible antagonism between the SVD basis where the inversion of the system is easy, and a 'localised' basis where the signal is sparsely represented, will be the issue we are trying to address here.

1.3. Cut-off, linear methods, thresholding. *The reader may profitably look at Subsections* 5.3 *and* 5.4*, where the linear methods and thresholding techniques are presented in detail in the direct case.*

SVD as well as Galerkin methods are very sensitive with respect to the choice of the tuning parameter $N(\varepsilon)$. This problem can be solved theoretically. However the solution heavily depends on prior assumptions of regularity on the solution, which have to be known in advance.

In the last ten years, many nonlinear methods have been developed especially in the direct case with the objective of automatically adapting to the unknown smoothness and local singular behavior of the solution. In the direct case, one of the most attractive methods is probably wavelet thresholding, since it allies numerical simplicity to asymptotic optimality on a large variety of functional classes such as Besov or Sobolev classes.

To adapt this approach in inverse problems, Donoho [11] introduced a wavelet-like decomposition, specifically adapted to the operator K (wavelet–vaguelette-decomposition) and provided a thresholding algorithm on this decomposition. In Abramovitch and Silverman [1], this method was compared with the similar vaguelette–wavelet-decomposition. Other wavelet approaches, might be mentioned such as Antoniadis and Bigot [2], Antoniadis et al. [3] and, especially for the deconvolution problem, Penski and Vidakovic [37], Fan and Koo [17], Kalifa and Mallat [24], Neelamani et al. [34].

Later, Cohen et al. [8] introduced an algorithm combining a Galerkin inversion with a thresholding algorithm.

The approach developed in the sequel is greatly influenced by these previous works. The accent we put here is on constructing (when necessary) new generation wavelet-type bases well adapted to the operator K, instead of sticking to the standard wavelet bases and reducing the range of potential operators covered by the method.

2. Wave-VD-type estimation

We explain here the basic idea of the method, which is very simple. Let us expand f using a well-suited basis ('the wavelet-type' basis', to be defined later):

$$f = \sum (f, \psi_\lambda)_{\mathbb{H}} \psi_\lambda.$$

Using Parseval's identity we have $\beta_\lambda = (f, \psi_\lambda)_{\mathbb{H}} = \sum f_i \psi_\lambda^i$ for $f_i = (f, e_i)_{\mathbb{H}}$ and $\psi_\lambda^i = (\psi_\lambda, e_i)_{\mathbb{H}}$. Let us put $Y_i = (Y_\varepsilon, g_i)_{\mathbb{K}}$. We then have

$$Y_i = (Kf, g_i)_{\mathbb{K}} + \varepsilon\xi_i = (f, K^* g_i)_{\mathbb{K}} + \varepsilon\xi_i = \Big(\sum_j f_j e_j, K^* g_i\Big)_{\mathbb{H}} + \varepsilon\xi_i = b_i f_i + \varepsilon\xi_i,$$

where the ξ_i's are forming a sequence of independent centered gaussian variables with variance 1. Furthermore,

$$\hat{\beta}_\lambda = \sum_i \frac{Y_i}{b_i} \psi_\lambda^i$$

is such that $\mathbb{E}(\hat{\beta}_\lambda) = \beta_\lambda$ (i.e. its average value is β_λ). It is a plausible estimate of β_λ. Let us now put ourselves in a multiresolution setting, taking $\lambda = (j, k)$ for $j \geq 0$, k belonging to a set χ_j, and consider

$$\hat{f} = \sum_{j=-1}^{J} \sum_{k \in \chi_j} t(\hat{\beta}_{jk}) \psi_{jk},$$

where t is a thresholding operator. (*The reader who is unfamiliar with thresholding techniques is referred to Section* 5.4.)

$$t(\hat{\beta}_{jk}) = \hat{\beta}_{jk} I\{|\hat{\beta}_{jk}| \geq \kappa t_\varepsilon \sigma_j\}, \quad t_\varepsilon = \varepsilon\sqrt{\log 1/\varepsilon}, \tag{6}$$

where $I\{A\}$ denotes the indicator function of the set A^*. Here κ is a tuning parameter of the method which will be properly chosen later. A main difference here with the direct case is the fact that the thresholding is depending on the resolution level through the constant σ_j which also will be stated more precisely later. Our main discussion will concern the choice of the basis (ψ_{jk}). In particular, we shall see that coherence properties with the SVD basis are of special interest.

We will particularly focus on two situations (corresponding to the two examples discussed in the introduction). In the first type of cases, the operator has as SVD bases the Fourier basis. In this case, this 'coherence' is easily obtained with 'standard' wavelets (still, not any kind of standard wavelet as will be seen). However, more difficult problems (and typically Wicksell's problem) require, when we need to mix these coherence conditions with the desired property of localisation of the basis, the construction of new objects: second generation-type wavelets.

2.1. WAVE-VD in a wavelet scenario. In this section we take $\{\psi_{jk},\ j \geq -1,\ k \in \chi_j\}$ to be a standard wavelet basis. More precisely, we suppose as usual that ψ_{-1} stands for the scaling function and, for any $j \geq -1$, χ_j is a set of order 2^j contained in $\mathbb{N}$. Moreover, we assume that the following properties are true. There exist constants c_p, C_p, d_p such that

$$c_p 2^{j(\frac{p}{2}-1)} \leq \|\psi_{jk}\|_p^p \leq C_p 2^{j(\frac{p}{2}-1)}, \tag{7}$$

$$\Big\| \sum_{k \in \chi_j} u_k \psi_{jk} \Big\|_p^p \leq D_p \sum_{k \in \chi_j} |u_k^p| \|\psi_{jk}\|_p^p \quad \text{for any sequence } u_k. \tag{8}$$

It is well known (see for instance Meyer [32]) that wavelet bases provide characterisations of smoothness spaces such as Hölder spaces Lip(s), Sobolev spaces W_p^s as well as Besov spaces B_{pq}^s for a range of indices s depending on the wavelet ψ. For the scale of Besov spaces which includes as particular cases Lip$(s) = B_{\infty\infty}^s$ (if $s \notin \mathbb{N}$) and $W_p^s = B_{pp}^s$ (if $p = 2$), the characterisation has the following form:

$$\text{If } f = \sum_{j \geq -1} \sum_{k \in \mathbb{Z}} \beta_{jk} \psi_{jk}, \text{ then } \|f\|_{B_{pq}^s} \sim \Big\| \big(2^{j[s+\frac{1}{2}-\frac{1}{p}]} \|\beta_{j\cdot}\|_{l_p}\big)_{j \geq -1} \Big\|_{l_q}. \tag{9}$$

As in Section 5, we consider the loss of a decision $\hat{f}$ if the truth is f as the $\mathbb{L}_p$ norm $\|\hat{f} - f\|_p$, and its associated risk

$$\mathbb{E}\|\hat{f} - f\|_p^p.$$

Here $\mathbb{E}$ denotes the expectation with respect to the random part of the observation y_ε. The following theorem is going to evaluate this risk, when the strategy is the one introduced in the previous section, and when the true function belongs to a Besov ball ($f \in B^s_{\pi,r}(M) \iff \|f\|_{B^s_{pq}} \le M$). One nice property of this estimation procedure is that it does not need the a priori knowledge of this regularity to get a good rate of convergence. If (e_k) is the SVD basis introduced in Section 1.2, b_k are the singular values and $\psi^i_{jk} = \langle e_i, \psi_{jk}\rangle$, we consider the estimator $\hat{f}$ defined in the beginning of Section 2.

Theorem 2.1. *Assume that* $1 < p < \infty$, $2\nu + 1 > 0$ *and*

$$\sigma_j^2 := \sum_i \left[\frac{\psi^i_{jk}}{b_i}\right]^2 \le C 2^{2j\nu} \quad \textit{for all } j \ge 0. \tag{10}$$

Put $\kappa^2 \ge 16p$, $2^J = [t_\varepsilon]^{\frac{-2}{2\nu+1}}$. *If* f *belongs to* $B^s_{\pi,r}(M)$ *with* $\pi \ge 1$, $s \ge 1/\pi$, $r \ge 1$ (*with the restriction* $r \le \pi$ *if* $s = (2\nu+1)\big(\frac{p}{2\pi} - \frac{1}{2}\big)$), *then we have*

$$\mathbb{E}\|\hat{f} - f\|_p^p \le C \log(1/\varepsilon)^{p-1} [\varepsilon^2 \log(1/\varepsilon)]^{\alpha p}, \tag{11}$$

with

$$\alpha = \frac{s}{1 + 2(\nu + s)} \qquad \textit{if } s \ge (2\nu+1)\Big(\frac{p}{2\pi} - \frac{1}{2}\Big),$$

$$\alpha = \frac{s - 1/\pi + 1/p}{1 + 2(\nu + s - 1/\pi)} \quad \textit{if } \frac{1}{\pi} \le s < (2\nu+1)\Big(\frac{p}{2\pi} - \frac{1}{2}\Big).$$

Remarks. 1. Condition (10) is essential here. As will be shown later, this condition is linking the wavelet system with the singular value decomposition of the kernel K. If we set ourselves in the deconvolution case, the SVD basis is the *Fourier* basis in such a way that ψ^i_{jk} is simply the Fourier coefficient of ψ_{jk}. If we choose as wavelet basis the periodized Meyer wavelet basis (see Meyer [32] and Mallat [30]), conditions (7) and (8) are satisfied. In addition, as the Meyer wavelet has the remarkable property of being compactly supported in the Fourier domain, simple calculations prove that, for any $j \ge 0$, k, the number of i's such that $\psi^i_{jk} \ne 0$ is finite and equal to 2^j. Then if we assume to be in the so-called 'regular' case ($b_k \sim k^{-\nu}$, for all k), it is easy to establish that (10) is true. This condition is also true for more general cases in the deconvolution setting such as the box-car deconvolution, see [22], [27].

2. These results are minimax (see [43]) up to logarithmic factors. This means that if we consider the best estimator in its worst performance over a given Besov

ball, this estimator attains a rate of convergence which is the one given in (11) up to logarithmic factors.

3. If we compare these results with the rates of convergence obtained in the direct model (see Subsections 5.3 and 5.4), we see that the difference (up to logarithmic terms) essentially lies in the parameter ν which acts as a reducing factor of the rate of convergence. This parameter quantifies the extra difficulty offered by the inverse problem. It is often called coefficient of illposedness. If we recall that in the deconvolution case, the coefficients b_k are the Fourier coefficients of the function γ, the illposedness coefficient then clearly appears to be closely related to the regularity of the blurring function.

This result has been proved in the deconvolution case in [22]. The proof of the theorem is given in Appendix I.

2.2. WAVE-VD in Jacobi scenario: NEED-VD. We have seen that the results given above are true under the condition (10) on the wavelet basis.

Let us first appreciate how the condition (10) links the 'wavelet-type' basis to the SVD basis (e_k). To see this let us put ourselves in the regular case:

$$b_i \sim i^{-\nu}.$$

(By this we mean more precisely that there exist two positive constants c and c' such that $c' i^{-\nu} \le b_i \le c i^{-\nu}$.)

If (10) is true, we have

$$C 2^{2j\nu} \ge \sum_m \sum_{2^m \le i < 2^{m+1}} \left[\frac{\psi_{jk}^i}{b_i} \right]^2.$$

Hence, for all $m \ge j$,

$$\sum_{2^m \le i < 2^{m+1}} [\psi_{jk}^i]^2 \le c 2^{2\nu(j-m)}.$$

This suggests the necessity to construct a 'wavelet-type' basis having support, at the level j, with respect to the SVD basis (sum in i) concentrated on the integers between 2^j and 2^{j+1} and exponentially decreasing after this band. This is exactly the case of Meyer's wavelet, when the SVD basis is the Fourier basis.

In the general case of an arbitrary linear operator giving rise to an arbitrary SVD basis (e_k), and if in addition to (10) we add a localisation condition on the basis, we do not know if such a construction can be performed. However, in some cases, even quite as far from the deconvolution as the Wicksell problem, one can build a 'second generation wavelet-type' basis, with exactly these properties.

The following construction due to Petrushev and collaborators ([33], [39], [38]) exactly realizes the paradigm mentioned above, producing a frame (the needlet basis) in the case of Jacobi polynomials (as well as in different other cases such as spherical harmonics, Hermite functions, Laguerre polynomials) which has the property of being localised.

3. Petrushev construction of needlets

Frames were introduced in the 1950s by Duffin and Schaeffer [15] to represent functions via over-complete sets. Frames including tight frames arise naturally in wavelet analysis on $\mathbb{R}^d$. Tight frames which are very close to orthonormal bases are especially useful in signal and image processing.

We will see that the following construction has the advantage of being easily computable and producing well-localised tight frames constructed on a specified orthonormal basis.

We recall the following definition.

Definition 3.1. Let $\mathbb{H}$ be a Hilbert space. A sequence (e_n) in $\mathbb{H}$ is said to be a *tight frame* (with constant 1) if

$$\|f\|^2 = \sum_n |\langle f, e_n\rangle|^2 \quad \text{for all } f \in \mathbb{H}.$$

Let now $\mathcal{Y}$ be a metric space, μ a finite measure. Let us suppose that we have the decomposition

$$\mathbb{L}_2(\mathcal{Y}, \mu) = \bigoplus_{k=0}^{\infty} H_k,$$

where the H_k's are finite dimensional spaces. For the sake of simplicity we suppose that H_0 is reduced to the constants.

Let L_k be the orthogonal projection on H_k:

$$L_k(f)(x) = \int_{\mathcal{Y}} f(y) L_k(x, y) d\mu(y) \quad \text{for all } f \in \mathbb{L}_2(\mathcal{Y}, \mu),$$

where

$$L_k(x, y) = \sum_{i=1}^{l_k} e_i^k(x) \bar{e}_i^k(y),$$

l_k is the dimension of H_k and $(e_i^k)_{i=1,\dots,l_k}$ is an orthonormal basis of H_k. Observe that we have the following property of the projection operators:

$$\int L_k(x, y) L_m(y, z) d\mu(z) = \delta_{k,m} L_k(x, z). \tag{12}$$

The construction, also inspired by the paper of Frazier, Jawerth and Weiss [18], is based on two fundamental steps: Littlewood–Paley decomposition and discretization, which are summarized in the following two subsections.

3.1. Littlewood–Paley decomposition. Let φ be a C^∞ function supported in $|\xi| \leq 1$ such that $1 \geq \varphi(\xi) \geq 0$ and $\varphi(\xi) = 1$ if $|\xi| \leq \frac{1}{2}$. We define

$$a^2(\xi) = \varphi(\xi/2) - \varphi(\xi) \geq 0$$

so that

$$\sum_j a^2(\xi/2^j) = 1 \quad \text{for all } |\xi| \geq 1. \tag{13}$$

We further define the operator

$$\Lambda_j = \sum_{k \geq 0} a^2(k/2^j) L_k$$

and the associated kernel

$$\Lambda_j(x, y) = \sum_{k \geq 0} a^2(k/2^j) L_k(x, y) = \sum_{2^{j-1} < k < 2^{j+1}} a^2(k/2^j) L_k(x, y).$$

The following assertion is true.

Proposition 3.2. *For all* $f \in \mathbb{H}$

$$f = \lim_{J \to \infty} L_0(f) + \sum_{j=0}^{J} \Lambda_j(f) \tag{14}$$

and

$$\Lambda_j(x, y) = \int M_j(x, z) M_j(z, y) d\mu(z) \quad \textit{for } M_j(x, y) = \sum_k a(k/2^j) L_k(x, y). \tag{15}$$

Proof.

$$L_0(f) + \sum_{j=0}^{J} \Lambda_j(f) = L_0 + \sum_{j=0}^{J} \Big(\sum_k a^2(k/2^j) L_k \Big) = \sum_k \varphi(k/2^{J+1}) L_k \tag{16}$$

Hence

$$\begin{aligned} &\Big\| \sum_k \varphi(k/2^{J+1}) L_k(f) - f \Big\|^2 \\ &\quad = \sum_{l \geq 2^{J+1}} \|L_l(f)\|^2 + \sum_{2^J \leq l < 2^{J+1}} \|L_l(f)(1 - \varphi(l/2^{J+1})\|^2 \\ &\quad \leq \sum_{l \geq 2^J} \|L_l(f)\|^2 \longrightarrow 0, \quad \text{when } J \to \infty. \end{aligned}$$

(15) is a simple consequence of (12). □

3.2. Discretization. Let us define

$$\mathcal{K}_k = \bigoplus_{m=0}^{k} H_m,$$

and let us assume that some additional assumptions are true:

1. $f \in \mathcal{K}_k \Longrightarrow \bar{f} \in \mathcal{K}_k$.
2. $f \in \mathcal{K}_k, g \in \mathcal{K}_l \Longrightarrow fg \in \mathcal{K}_{k+l}$.
3. Quadrature formula: for all $k \in \mathbb{N}$, there exists a finite subset χ_k of $\mathcal{Y}$ and positive real numbers $\lambda_\xi > 0$ indexed by the elements ξ of χ_k such that

$$\int f\, d\mu = \sum_{\xi \in \chi_k} \lambda_\xi f(\xi) \quad \text{for all } f \in \mathcal{K}_k.$$

Then the operator M_j defined in the subsection above is such that $M_j(x, z) = \overline{M_j(z, x)}$ and

$$z \mapsto M_j(x, z) \in \mathcal{K}_{2^{j+1}-1}.$$

Hence

$$z \mapsto M_j(x, z) M_j(z, y) \in \mathcal{K}_{2^{j+2}-2},$$

and we can write

$$\Lambda_j(x, y) = \int M_j(x, z) M_j(z, y)\, d\mu(z) = \sum_{\xi \in \chi_{2^{j+2}-2}} \lambda_\xi M_j(x, \xi) M_j(\xi, y).$$

This implies

$$\begin{aligned} \Lambda_j f(x) &= \int \Lambda_j(x, y) f(y)\, d\mu(y) = \int \sum_{\xi \in \chi_{2^{j+2}-2}} \lambda_\xi M_j(x, \xi) M_j(\xi, y) f(y)\, d\mu(y) \\ &= \sum_{\xi \in \chi_{2^{j+2}-2}} \sqrt{\lambda_\xi} M_j(x, \xi) \int \overline{\sqrt{\lambda_\xi} M_j(y, \xi)} f(y)\, d\mu(y). \end{aligned}$$

This can be summarized in the following way if we put $\sqrt{\lambda_\xi} M_j(x, \xi) = \psi_{j,\xi}(x)$ and $\chi_{2^{j+2}-2} = \mathbb{Z}_j$:

$$\Lambda_j f(x) = \sum_{\xi \in \mathbb{Z}_j} \langle f, \psi_{j,\xi} \rangle \psi_{j,\xi}(x).$$

Proposition 3.3. *The family $(\psi_{j,\xi})_{j \in \mathbb{N}, \xi \in \mathbb{Z}_j}$ is a tight frame.*

Proof. As

$$f = \lim_{J\to\infty} (L_0(f) + \sum_{j\leq J} \Lambda_j(f)),$$

we have

$$\|f\|^2 = \lim_{J\to\infty} (\langle L_0(f), f\rangle + \sum_{j\leq J} \langle \Lambda_j(f), f\rangle),$$

but

$$\langle \Lambda_j(f), f\rangle = \sum_{\xi\in\mathbb{Z}_j} \langle f, \psi_{j,\xi}\rangle\langle \psi_{j,\xi}, f\rangle = \sum_{\xi\in\mathbb{Z}_j} |\langle f, \psi_{j,\xi}\rangle|^2,$$

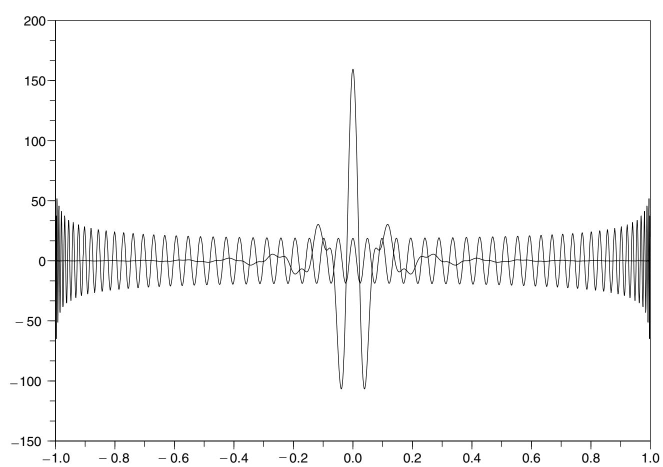
and if ψ_0 is a normalized constant we have $\langle L_0(f), f\rangle = |\langle f, \psi_0\rangle|^2$ so that

$$\|f\|^2 = |\langle f, \psi_0\rangle|^2 + \sum_{j\in\mathbb{N},\ \xi\in\mathbb{Z}_j} |\langle f, \psi_{j,\xi}\rangle|^2.$$

But this is exactly the characterization of a tight frame. □

3.3. Localisation properties. This construction has been performed in different frameworks by Petrushev and coauthors giving in each situation very nice localisation properties.

The following figure (thanks to Paolo Baldi) is an illustration of this phenomenon: it shows a needlet constructed as explained above using Legendre polynomials of degree 2^8. The highly oscillating function is a Legendre polynomial of degree 2^8, whereas the localised one is a needlet centered approximately in the middle of the interval. Its localisation properties are remarkable considering the fact that both functions are polynomials of the same order.



In the case of the sphere of $\mathbb{R}^{d+1}$, where the spaces H_k are spanned by spherical harmonics, the following localisation property is proved in Narcowich, Petrushev and Ward [33]: for any k there exists a constant C_k such that

$$|\psi_{j\eta}(\xi)| \leq \frac{C_k 2^{dj/2}}{[1+2^j \arccos < \eta, \xi >]^k}.$$

A similar result exists in the case of Laguerre polynomials on $\mathbb{R}_+$ [25].

In the case of Jacobi polynomials on the interval with Jacobi weight, the following localisation property is shown by Petrushev and Xu [38]. For any k there exist constants C, c such that

$$|\psi_{j\eta}(\cos\theta)| \leq \frac{C 2^{j/2}}{(1+(2^j|\theta - \arccos \eta|)^k \sqrt{w_{\alpha\beta}(2^j, \cos\theta)}},$$

where $w_{\alpha\beta}(n,x) = (1-x+n^{-2})^{\alpha+1/2}(1+x+n^{-2})^{\beta+1/2}, -1 \leq x \leq 1$ if $\alpha > -1/2$, $\beta > -1/2$.

4. NEED-VD in the Jacobi case

Let us now come back to the estimation algorithm.

We consider the inverse problem (2) with

$$\mathbb{H} = \mathbb{L}_2(I, d\gamma(x)), \quad I = [-1,1], \quad d\gamma(x) = \omega_{\alpha,\beta}(x)dx,$$
$$\omega_{\alpha,\beta}(x) = (1-x)^\alpha (1+x)^\beta; \quad \alpha > -1/2, \ \beta > -1/2.$$

For the sake of simplicity, let us suppose $\alpha \geq \beta$. (Otherwise we can exchange the parameters.)

Let P_k be the normalized Jacobi polynomial for this weight. We suppose that these polynomials appear as SVD basis of the operator K, as it is the case for the Wicksell problem with $\beta = 0$, $\alpha = 1$, $b_k \sim k^{-1/2}$.

4.1. Needlets and condition (10). Let us define the 'needlets' as constructed above:

$$\psi_{j,\eta_k}(x) = \sum_l \hat{a}(l/2^{j-1}) P_l(x) P_l(\eta_k) \sqrt{b_{j,\eta_k}}. \tag{17}$$

The following proposition asserts that such a construction always implies the condition (10) in the regular case.

Proposition 4.1. *Assume that ψ_{j,η_k} is a frame. If $b_i \sim i^{-\nu}$ then*

$$\sigma_j^2 := \sum_i \left[\frac{\psi_{jk}^i}{b_i}\right]^2 \leq C 2^{2j\nu}.$$

Proof. Suppose the family ψ_{j,η_k} is a frame (not necessarily tight). As the elements of a frame are bounded and the set $\{i,\ \psi^i_{jk} \neq 0\}$ is included in the set $\{C_1 2^j, \dots, C_2 2^j\}$, we have

$$\sum_i \left[\frac{\psi^i_{jk}}{b_i}\right]^2 \leq C 2^{j\nu} \|\psi_{j,\eta_k}\|^2 \leq C' 2^{j\nu}. \qquad \square$$

4.2. Convergence results in the Jacobi case. The following theorem is the analogous of Theorem 2.1 in this case. As can be seen, the results there are at the same time more difficult to obtain (the following theorem does not cover the same range as the previous one) and richer since they furnish new rates of convergence.

Theorem 4.2. *Suppose that we are in the Jacobi case as stated above* ($\alpha \geq \beta > -\frac{1}{2}$).
We put

$$t_\varepsilon = \varepsilon\sqrt{\log 1/\varepsilon},$$
$$2^J = t_\varepsilon^{-\frac{2}{1+2\nu}},$$

choose $\kappa \geq 16p\big[1 + \big(\frac{\alpha}{2} - \frac{\alpha+1}{p}\big)_+\big]$*, and suppose that we are in the regular case, i.e.*

$$b_i \sim i^{-\nu}, \quad \nu > -\frac{1}{2}.$$

Then, if $f = \sum_j \sum_k \beta_{j,\eta_k} \psi_{j,\eta_k}$ *is such that*

$$\Big(\sum |\beta_{j,\eta_k}|^p \|\psi_{j,\eta_k}\|_p^p\Big)^{1/p} \leq \rho_j 2^{-js}, \quad (\rho_j) \in l_r,$$

it follows that

$$\mathbb{E}\|\hat{f} - f\|_p^p \leq C[\log(1/\varepsilon)]^{p-1}\big[\varepsilon\sqrt{\log(1/\varepsilon)}\,\big]^{\mu p}$$

with

1. *if* $p < 2 + \frac{1}{\alpha+1/2}$*, then*
$$\mu = \frac{s}{s + \nu + \frac{1}{2}};$$
2. *if* $p > 2 + \frac{1}{\alpha+1/2}$*, then*
$$\mu = \frac{s}{s + \nu + \alpha + 1 - \frac{2(1+\alpha)}{p}}.$$

This theorem is proved in Kerkyacharian et al. [26]. Simulation results on these methods are given there, showing that their performances are far above the usual SVD methods in several cases. It is interesting to notice that the rates of convergence which are obtained here agree with the minimax rates evaluated in Johnstone and Silverman [23] where the case $p = 2$ is considered. But the second case ($p > 2 + \frac{1}{\alpha+1/2}$) shows a rate of convergence which is new in the literature. In [26], where the whole range of Besov bodies is considered, more atypical rates are given.

5. Direct models ($K = I$): a memento

5.1. The density model. The most famous nonparametric model consists in observing n i.i.d. random variables having a common density f on the interval [0, 1], and in trying to give an estimation of f.

A standard route to perform this estimation consists in expanding the density f in an orthonormal basis $\{e_k,\ k \in \mathbb{N}\}$ of a Hilbert space $\mathbb{H}$ – assuming implicitly that f belongs to $\mathbb{H}$:

$$f = \sum_{l \in \mathbb{N}} \theta_l e_l.$$

If $\mathbb{H}$ happens to be the space $\mathbb{L}_2 = \{g : [0, 1] \mapsto \mathbb{R},\ \|g\|_2^2 := \int_0^1 g^2 < \infty\}$, we observe that

$$\theta_l = \int_0^1 e_l(x) f(x) dx = \mathbb{E} e_l(X_i).$$

Replacing the expectation by the empirical one leads to a standard estimate for θ_l:

$$\hat{\theta}_l = \frac{1}{n} \sum_{i=1}^{n} e_l(X_i).$$

At this step, the simplest choice of estimate for f is obviously:

$$\hat{f}_m = \sum_{i=1}^{m} \hat{\theta}_l e_l. \tag{18}$$

5.2. From the density to the white noise model. Before analysing the properties of the estimator defined above, let us observe that the previous approach (representing f by its coefficients $\{\theta_k,\ k \geq 0\}$, leads to summarize the information in the following sequence model:

$$\{\hat{\theta}_k,\ k \geq 0\}. \tag{19}$$

We can write $\hat{\theta}_k =: \theta_k + u_k$, with

$$u_k = \frac{1}{n} \sum_{i=1}^{n} [e_k(X_i) - \theta_k],$$

The central limit theorem is a relatively convincing argument that the model (19) may be approximated by the following one:

$$\Big\{\hat{\theta}_k = \theta_k + \frac{\eta_k}{\sqrt{n}},\ k \geq 0\Big\}, \tag{20}$$

where the η_k's are forming a sequence of i.i.d. gaussian, centered variables with fixed variance σ^2, say. Such an approximation requires more delicate calculations than these quick arguments and is rigourously proved in Nussbaum [35], see also Brown and Low [5]. This model is the sequence space model associated to the following global observation, the so-called white noise model (with $\varepsilon = n^{-1/2}$):

$$dY_t = f(t)dt + \varepsilon dW_t, \quad t \in [0, 1],$$

where for any $\varphi \in \mathbb{L}^2([0, 1], dt)$, $\int_{[0,1]} \varphi(t)dY_t = \int_{[0,1]} f(t)\varphi(t)dt + \varepsilon \int_{[0,1]} \varphi(t)dW_t$ is observable.

(20) formally consists in considering all the observables obtained for $\varphi = e_k$ for all k in $\mathbb{N}$. Among nonparametric situations, the white noise model considered above is one of the simplest, at least technically. Mostly for this reason, this model has been given a central place in statistics, particularly by the Russian school, following Ibraguimov and Has'minskii (see for instance their book [20]). However it arises as an appropriate large sample limit to more general nonparametric models, such as regression with random design, or non independent spectrum estimation, diffusion models – see for instance [21], [4],. . . .

5.3. The linear estimation: how to choose the tuning parameter *m*? In (18), the choice of m is crucial.

To better understand the situation let us have a look at the risk of the strategy $\hat{f}_m$. If we consider that, when deciding $\hat{f}_m$ when f is the truth, we have a loss of order $\|\hat{f}_m - f\|_2^2$, then our risk will be the following mathematical expectation:

$$\mathbb{E}\|\hat{f}_m - f\|_2^2.$$

Of course this way of measuring our risk is arguable since there is no particular reason for the $\mathbb{L}_2$ norm to reflect well the features we want to recover in the signal. For instance, an $\mathbb{L}_\infty$-norm could be preferred because it is easier to visualize. In general, several $\mathbb{L}_p$ norms are considered (as it is the case in Sections 2.1 and 4.2). Here we restrict to the $\mathbb{L}_2$ case for sake of simplicity.

To avoid technical difficulties, we set ourselves in the case of a white noise model, considering that we observe the sequence defined in (20). Hence,

$$\mathbb{E}(\hat{\theta}_l - \theta_l)^2 = \frac{1}{n}\int_0^1 e_l(x)^2 dx = \frac{1}{n} := \varepsilon^2.$$

We are now able to obtain

$$\mathbb{E}\|\hat{f}_m - f\|_2^2 = \sum_{l \le m}(\hat{\theta}_l - \theta_l)^2 + \sum_{l>m}\theta_l^2 \le m\varepsilon^2 + \sum_{l>m}\theta_l^2.$$

Now assume that f belongs to the following specified compact set of l_2:

$$\sum_{l>k}\theta_l^2 \le Mk^{-2s} \quad \text{for all } k \in \mathbb{N}_*, \tag{21}$$

for some $s > 0$ which is here an index of regularity directly connected to the size of the compact set in l_2 containing the function f. Then we obtain

$$\mathbb{E}\|\hat{f}_m - f\|_2^2 \leq m\varepsilon^2 + Mm^{-2s}.$$

We observe that the RHS is the sum of two factors: one (called the stochastic term) is increasing in m and reflects the fact that because of the noise, the more coefficients we have to estimate, the larger the global error will be. The second one (called the bias term or approximation term) does not depend on the noise and is decreasing in m. The RHS is optimised by choosing $m = m_*(s) =: c(s, M)\varepsilon^{\frac{-2}{1+2s}}$. Then

$$\mathbb{E}\|\hat{f}_{m_*(s)} - f\|_2 \leq c'(s, M)\varepsilon^{\frac{-4s}{1+2s}}.$$

Let us observe that the more f is supposed to be regular (in the sense the larger s is), the less coefficients we need to estimate: a very irregular function (s close to 0) requires almost as much as $\varepsilon^{-2} = n$ coefficients, which corresponds to estimate as many coefficients as the number of available observations – in the density model for instance. The rate obtained in (5.3) can be proved to be optimal in the following sense (minimax): if we consider the best estimator in its worst performance over the class of functions verifying (21), this estimator attains a rate of convergence which is (up to a constant) the one given in (5.3). See Tsybakov [40] for a detailed review of the minimax point of view.

5.4. The thresholding estimation. Let us now suppose that the constant s, which plays an essential role in the construction of the previous estimator is not known. This is realistic, since it is extremely rare to know in advance that the function we are seeking has a specified regularity. Also, the previous approach takes very seriously into account the order in which the basis is taken. Let us now present a very elegant way of addressing at the same time both of these issues. The thresholding techniques which have been known for long by engineers in electronic and telecommunications, was introduced in statistics in Donoho and Johnstone [14] and later in a series of papers on wavelet thresholding [12], [13]. It allies numerical simplicity to asymptotic optimality.

It starts from a different kind of observation. Let us introduce the following estimate:

$$\widetilde{f} = \sum_{k=0}^{B} \hat{\theta}_k \mathbb{I}\{|\hat{\theta}_k| \geq \kappa t_\varepsilon\} e_k. \tag{22}$$

Here the point of view is the following. We choose B very large (i.e. almost corresponding to $s = 0$):

$$B = \varepsilon^{-2} \log 1/\varepsilon.$$

But instead of keeping all the coefficients θ_k such that k is between 0 and B, we decide to kill those which are not above the threshold t_ε. The intuitive justification of this choice is as follows. Assuming that f has some kind of regularity condition like (21) (unknown, but real...), essentially means that the coefficients θ_k of f are of small magnitude except perhaps a small number of them. Obviously, in the reconstruction of f, only the large coefficients will be significant. t_ε is chosen in such a way that the noise $\hat{\theta}_k - \theta_k$ due to the randomness of the observation might be neglected:

$$t_\varepsilon = \varepsilon[\log 1/\varepsilon]^{-1/2}.$$

Now let us assume another type of condition on f – easily interpreted by the fact that f is sparsely represented in the basis (e_k) – namely: there exists a positive constant $0 < q < 2$ such that

$$\sup_{\lambda>0} \lambda^q \#\{k,\ |\theta_k| \geq \lambda\} \leq M \quad \text{for all } k \in \mathbb{N}_*, \tag{23}$$

$$\begin{aligned}\mathbb{E}\|\widetilde{f} - f\|_2^2 &= \sum_{l\leq B}(\hat{\theta}_l \mathbb{I}\{|\hat{\theta}_l| \geq \kappa t_\varepsilon\} - \theta_l)^2 + \sum_{l>B}\theta_l^2 \\ &\leq \sum_l (\hat{\theta}_l - \theta_l)^2 \mathbb{I}\{|\theta_l| \geq \kappa t_\varepsilon/2\} + \sum_l \theta_l^2 \mathbb{I}\{|\theta_l| \leq 2\kappa t_\varepsilon\} \\ &\quad + \sum_{l\leq B}[(\hat{\theta}_l - \theta_l)^2 + \theta_l^2]\mathbb{I}\{|\hat{\theta}_l - \theta_l| \geq \kappa t_\varepsilon/2\} + \sum_{l>B}\theta_l^2.\end{aligned}$$

Now, using the probabilistic bounds

$$\mathbb{E}(\hat{\theta}_l - \theta_l)^2 = \varepsilon^2, \qquad \mathbb{P}(|\hat{\theta}_l - \theta_l| \geq \lambda) \leq 2\exp -\frac{\lambda^2}{2\varepsilon^2} \quad \text{for all } \lambda > 0,$$

and the fact that condition (23) implies

$$\sum_l \theta_l^2 \mathbb{I}\{|\theta_l| \leq 2\kappa t_\varepsilon\} \leq C t_\varepsilon^{2-q},$$

we get

$$\mathbb{E}\|\widetilde{f} - f\|_2^2 \leq M\varepsilon^2 t_\varepsilon^{-q} + C' t_\varepsilon^{2-q} + \varepsilon^{\kappa^2/8} B + \sum_{l>B}\theta_l^2.$$

It remains now to choose $\kappa^2 \geq 32$ in order to get

$$\mathbb{E}\|\widetilde{f} - f\|_2^2 \leq C' t_\varepsilon^{2-q} + \sum_{l>B}\theta_l^2,$$

and if we assume in addition to (23) that

$$\sum_{l>k}\theta_l^2 \leq Mk^{-\frac{2-q}{2}} \quad \text{for all } k \in \mathbb{N}_*, \tag{24}$$

then we get

$$\mathbb{E}\|\widetilde{f}-f\|_2^2 \le C" t_\varepsilon^{2-q}$$

Note that the interesting point in this construction is that the regularity conditions imposed on the function f are *not known* by the statistician, since they do not enter into the construction of the procedure. This property is called adaptation.

Now, to compare with the previous section, let us take $q=\frac{2}{1+2s}$. It is not difficult to prove that as soon as f verifies (21), it automatically verifies (23) and (24). Hence $\widetilde{f}$ and $\hat{f}_{m*(s)}$ have the same rate of convergence up to a logarithmic term. If we neglect this logarithmic loss, we substantially gain here the fact that we need not know the apriori regularity conditions on the aim function. It can also be proved that in fact conditions (23) and (24) are defining a set which is substantially larger than the set defined by condition (21): for instance its entropy is strictly larger (see [28]).

6. Appendix: Proof of Theorem 2.1

In this proof, C will denote an absolute constant which may change from one line to the other.

We can always suppose $p\ge\pi$. Indeed, if $\pi\ge p$ it is very simple to see that $B^s_{\pi,r}(M)$ is included into $B^s_{p,r}(M)$: as $2^{j[s+\frac12-\frac1p]}\|\beta_{j\cdot}\|_{l_p}\le 2^{j[s+\frac12-\frac1\pi]}\|\beta_{j\cdot}\|_{l_\pi}$ (since χ_j is of cardinality 2^j).

First we have the following decomposition:

$$\begin{aligned}\mathbb{E}\|\hat{f}-f\|_p^p &\le 2^{p-1}\Big\{\mathbb{E}\Big\|\sum_{j=-1}^{J}\sum_{k\in\chi_j}(t(\hat\beta_{jk})-\beta_{jk})\psi_{jk}\Big\|_p^p+\Big\|\sum_{j>J}\sum_{k\in\chi_j}\beta_{jk}\psi_{jk}\Big\|_p^p\Big\}\\ &=: \mathrm{I}+\mathrm{II}.\end{aligned}$$

The term II is easy to analyse: since f belongs to $B^s_{\pi,r}(M)$, using standard embedding results (which in this case simply follows from direct comparisons between l_q norms) we have that f also belong to $B^{s-(\frac1\pi-\frac1p)_+}_{p,r}(M')$, for some constant M'. Hence

$$\Big\|\sum_{j>J}\sum_{k\in\chi_j}\beta_{jk}\psi_{jk}\Big\|_p\le C2^{-J\underline{[}s-(\frac1\pi-\frac1p)_+]}.$$

Then we only need to verify that $\frac{s-(\frac1\pi-\frac1p)_+}{1+2\nu}$ is always larger that α, which is not difficult.

Bounding the term I is more involved. Using the triangular inequality together with Hölder's inequality and property (8) for the second line, we get

$$\mathrm{I} \le 2^{p-1} J^{p-1} \sum_{j=-1}^{J} \mathbb{E}\Big\| \sum_{k\in\chi_j} (t(\hat\beta_{jk}) - \beta_{jk})\psi_{jk} \Big\|_p^p$$

$$\le 2^{p-1} J^{p-1} D_p \sum_{j=-1}^{J} \sum_{k\in\chi_j} \mathbb{E}|t(\hat\beta_{jk}) - \beta_{jk}|^p \|\psi_{jk}\|_p^p.$$

Now, we separate four cases:

$$\sum_{j=-1}^{J} \sum_{k\in\chi_j} \mathbb{E}|t(\hat\beta_{jk}) - \beta_{jk}|^p \|\psi_{jk}\|_p^p$$

$$= \sum_{j=-1}^{J} \sum_{k\in\chi_j} \mathbb{E}|t(\hat\beta_{jk}) - \beta_{jk}|^p \|\psi_{jk}\|_p^p \big\{ I\{|\hat\beta_{jk}| \ge \kappa t_\varepsilon \sigma_j\} + I\{|\hat\beta_{jk}| < \kappa t_\varepsilon \sigma_j\} \big\}$$

$$\le \sum_{j=-1}^{J} \sum_{k\in\chi_j} \Big[\mathbb{E}|\hat\beta_{jk} - \beta_{jk}|^p \|\psi_{jk}\|_p^p I\{|\hat\beta_{jk}| \ge \kappa t_\varepsilon \sigma_j\}$$
$$\Big\{ I\Big\{|\beta_{jk}| \ge \frac{\kappa}{2} t_\varepsilon \sigma_j\Big\} + I\Big\{|\beta_{jk}| < \frac{\kappa}{2} t_\varepsilon \sigma_j\Big\}\Big\}$$
$$+ |\beta_{jk}|^p \|\psi_{jk}\|_p^p I\{|\hat\beta_{jk}| \le \kappa t_\varepsilon \sigma_j\}$$
$$\big\{ I\{|\beta_{jk}| \ge 2\kappa t_\varepsilon \sigma_j\} + I\{|\beta_{jk}| < 2\kappa t_\varepsilon \sigma_j\}\big\}\Big]$$

$$\le: Bb + Bs + Sb + Ss.$$

Notice that $\hat\beta_{jk} - \beta_{jk} = \sum_i \frac{Y_i - b_i f_i}{b_i} \psi_{jk}^i = \varepsilon \sum_i \xi_i \frac{\psi_{jk}^i}{b_i}$ is a centered gaussian random variable with variance $\varepsilon^2 \sum_i \big[\frac{\psi_{jk}^i}{b_i}\big]^2$. Also recall that we set $\sigma_j^2 =: \sum_i [\frac{\psi_{jk}^i}{b_i}]^2 \le C 2^{2j\nu}$ and denote by s_q the qth absolute moment of the gaussian distribution when centered and with variance 1. Then, using standard properties of the gaussian distribution, for any $q \ge 1$ we have

$$\mathbb{E}|\hat\beta_{jk} - \beta_{jk}|^q \le s_q \sigma_j^q \varepsilon^q, \quad \mathbb{P}\{|\hat\beta_{jk} - \beta_{jk}| \ge \frac{\kappa}{2} t_\varepsilon \sigma_j\} \le 2\varepsilon^{\kappa^2/8}.$$

Hence

$$Bb \le \sum_{j=-1}^{J} \sum_{k\in\chi_j} s_p \sigma_j^p \varepsilon^p \|\psi_{jk}\|_p^p I\{|\beta_{jk}| \ge \frac{\kappa}{2} t_\varepsilon \sigma_j\},$$

$$Ss \le \sum_{j=-1}^{J} \sum_{k\in\chi_j} |\beta_{jk}|^p \|\psi_{jk}\|_p^p I\{|\beta_{jk}| < 2\kappa t_\varepsilon \sigma_j\}$$

and

$$
\begin{aligned}
Bs &\le \sum_{j=-1}^{J}\sum_{k\in\chi_j}[\mathbb{E}|\hat{\beta}_{jk}-\beta_{jk}|^{2p}]^{1/2}\Big[\mathbb{P}\Big\{|\hat{\beta}_{jk}-\beta_{jk}|\ge\frac{\kappa}{2}t_\varepsilon\sigma_j\Big\}\Big]^{1/2}\\
&\qquad\qquad\qquad\qquad \|\psi_{jk}\|_p^p I\Big\{|\beta_{jk}|<\frac{\kappa}{2}t_\varepsilon\sigma_j\Big\}\\
&\le \sum_{j=-1}^{J}\sum_{k\in\chi_j} s_{2p}^{1/2}\sigma_j^p\varepsilon^p 2^{1/2}\varepsilon^{\kappa^2/16}\|\psi_{jk}\|_p^p I\Big\{|\beta_{jk}|<\frac{\kappa}{2}t_\varepsilon\sigma_j\Big\}\\
&\le C\sum_{j=-1}^{J}2^{jp(\nu+\frac{1}{2})}\varepsilon^p\varepsilon^{\kappa^2/16}\le C\varepsilon^{\kappa^2/16}.
\end{aligned}
$$

Now, if we remark that the β_{jk}'s are necessarily all bounded by some constant (depending on M) since f belongs to $B^s_{\pi,r}(M)$, and using (7),

$$
\begin{aligned}
Sb &\le \sum_{j=-1}^{J}\sum_{k\in\chi_j}|\beta_{jk}|^p\|\psi_{jk}\|_p^p\mathbb{P}\{|\hat{\beta}_{jk}-\beta_{jk}|\ge 2\kappa t_\varepsilon\sigma_j\}I\{|\beta_{jk}|\ge 2\kappa t_\varepsilon\sigma_j\}\\
&\le \sum_{j=-1}^{J}\sum_{k\in\chi_j}|\beta_{jk}|^p\|\psi_{jk}\|_p^p 2\varepsilon^{\kappa^2/8}I\{|\beta_{jk}|\ge 2\kappa t_\varepsilon\sigma_j\}\\
&\le C\sum_{j=-1}^{J}2^{j\frac{p}{2}}\varepsilon^{\kappa^2/8}\le C\varepsilon^{\frac{\kappa^2}{8}-\frac{p}{2(2\nu+1)}}.
\end{aligned}
$$

It is easy to check that in all cases, if $\kappa^2\ge 16p$ the terms Bs and Sb are smaller than the rates given in the theorem.

Using (7) and condition (10), for any $z\ge 0$ we have

$$
\begin{aligned}
Bb &\le C\varepsilon^p\sum_{j=-1}^{J}2^{j(\nu p+\frac{p}{2}-1)}\sum_{k\in\chi_j}I\Big\{|\beta_{jk}|\ge\frac{\kappa}{2}t_\varepsilon\sigma_j\Big\}\\
&\le C\varepsilon^p\sum_{j=-1}^{J}2^{j(\nu p+\frac{p}{2}-1)}\sum_{k\in\chi_j}|\beta_{jk}|^z[t_\varepsilon\sigma_j]^{-z}\\
&\le C{t_\varepsilon}^{p-z}\sum_{j=-1}^{J}2^{j[\nu(p-z)+\frac{p}{2}-1]}\sum_{k\in\chi_j}|\beta_{jk}|^z.
\end{aligned}
$$

Also, for any $p \geq z \geq 0$,

$$Ss \leq C \sum_{j=-1}^{J} 2^{j(\frac{p}{2}-1)} \sum_{k \in \chi_j} |\beta_{jk}|^z \sigma_j^{p-z} [t_\varepsilon]^{p-z}$$
$$\leq C[t_\varepsilon]^{p-z} \sum_{j=-1}^{J} 2^{j(\nu(p-z)+\frac{p}{2}-1)} \sum_{k \in \chi_j} |\beta_{jk}|^z.$$

So in both cases we have the same bound to investigate. We will write this bound in the following form (forgetting the constant):

$$\mathrm{I} + \mathrm{II} = t_\varepsilon{}^{p-z_1} \Big[\sum_{j=-1}^{j_0} 2^{j[\nu(p-z_1)+\frac{p}{2}-1]} \sum_{k \in \chi_j} |\beta_{jk}|^{z_1} \Big]$$
$$+ t_\varepsilon{}^{p-z_2} \Big[\sum_{j=j_0+1}^{J} 2^{j[\nu(p-z_2)+\frac{p}{2}-1]} \sum_{k \in \chi_j} |\beta_{jk}|^{z_2} \Big].$$

The constants z_i and j_0 will be chosen depending on the cases.

Let us first consider the case where $s \geq \big(\nu + \frac{1}{2}\big)\big(\frac{p}{\pi} - 1\big)$. Put

$$q = \frac{p(2\nu + 1)}{2(s + \nu) + 1}$$

and observe that, on the considered domain, $q \leq \pi$ and $p > q$. In the sequel it will be used that we automatically have $s = \big(\nu + \frac{1}{2}\big)\big(\frac{p}{q} - 1\big)$. Taking $z_2 = \pi$ we get

$$\mathrm{II} \leq t_\varepsilon{}^{p-\pi} \Big[\sum_{j=j_0+1}^{J} 2^{j[\nu(p-\pi)+\frac{p}{2}-1]} \sum_{k \in \chi_j} |\beta_{jk}|^{\pi} \Big].$$

Now, as

$$\frac{p}{2q} - \frac{1}{\pi} + \nu \left(\frac{p}{q} - 1 \right) = s + \frac{1}{2} - \frac{1}{\pi}$$

and

$$\sum_{k \in \chi_j} |\beta_{jk}|^{\pi} = 2^{-j(s+\frac{1}{2}-\frac{1}{\pi})} \tau_j$$

with $(\tau_j)_j \in l_r$ (this is a consequence of the fact that $f \in B^s_{\pi,r}(M)$ and (6)), we can write

$$\mathrm{II} \leq t_\varepsilon{}^{p-\pi} \sum_{j=j_0+1} 2^{jp(1-\frac{\pi}{q})(\nu+\frac{1}{2})} \tau_j^{\pi}$$
$$\leq C t_\varepsilon{}^{p-\pi} 2^{j_0 p(1-\frac{\pi}{q})(\nu+\frac{1}{2})}.$$

The last inequality is true for any $r \geq 1$ if $\pi > q$ and for $r \leq \pi$ if $\pi = q$. Notice that $\pi = q$ is equivalent to $s = (\nu + \frac{1}{2})(\frac{p}{\pi} - 1)$. Now if we choose j_0 such that $2^{j_0 \frac{p}{q}(\nu + \frac{1}{2})} \sim {t_\varepsilon}^{-1}$ we get the bound

$$ {t_\varepsilon}^{p-q} $$

which exactly gives the rate asserted in the theorem for this case.

As for the first part of the sum (before j_0), we have, taking now $z_1 = \widetilde{q}$, with $\widetilde{q} \leq \pi$, so that $\big[\frac{1}{2^j}\sum_{k\in\chi_j}|\beta_{jk}|^{\widetilde{q}}\big]^{\frac{1}{\widetilde{q}}} \leq \big[\frac{1}{2^j}\sum_{k\in\chi_j}|\beta_{jk}|^{\pi}\big]^{\frac{1}{\pi}}$, and using again (6),

$$
\begin{aligned}
\mathrm{I} &\leq {t_\varepsilon}^{p-\widetilde{q}}\Big[\sum_{-1}^{j_0} 2^{j[\nu(p-\widetilde{q})+\frac{p}{2}-1]}\sum_{k\in\chi_j}|\beta_{jk}|^{\widetilde{q}}\Big] \\
&\leq {t_\varepsilon}^{p-\widetilde{q}}\Big[\sum_{-1}^{j_0} 2^{j[\nu(p-\widetilde{q})+\frac{p}{2}-\frac{\widetilde{q}}{\pi}]}\sum_{k\in\chi_j}|\beta_{jk}|^{\pi}]^{\frac{\widetilde{q}}{\pi}}\Big] \\
&\leq {t_\varepsilon}^{p-\widetilde{q}}\sum_{-1}^{j_0} 2^{j[(\nu+\frac{1}{2})p(1-\frac{\widetilde{q}}{q})]}\tau_j^{\widetilde{q}} \\
&\leq C{t_\varepsilon}^{p-\widetilde{q}}2^{j_0[(\nu+\frac{1}{2})p(1-\frac{\widetilde{q}}{q})]} \\
&\leq C{t_\varepsilon}^{p-q}.
\end{aligned}
$$

The last two lines are valid if $\widetilde{q}$ is chosen strictly smaller than q (this is possible since $\pi \geq q$).

Let us now consider the case where $s < \big(\nu + \frac{1}{2}\big)\big(\frac{p}{q} - 1\big)$, and choose

$$ q = \frac{p}{2\big(s + \nu - \frac{1}{\pi}\big) + 1} $$

in such a way that we easily verify that $p - q = 2\frac{s-1/\pi+1/p}{1+2(\nu+s-1/\pi)}$, $q - \pi = \frac{(p-\pi)(1+2\nu)}{2\big(s+\nu-\frac{1}{\pi}\big)+1} > 0$, because s is supposed to be larger than $\frac{1}{\pi}$. Furthermore we also have $s + \frac{1}{2} - \frac{1}{\pi} = \frac{p}{2q} - \frac{1}{q} + \nu\big(\frac{p}{q} - 1\big)$.

Hence taking $z_1 = \pi$ and using again the fact that f belongs to $B^s_{\pi,r}(M)$,

$$
\begin{aligned}
\mathrm{I} &\leq {t_\varepsilon}^{p-\pi}\Big[\sum_{-1}^{j_0} 2^{j[\nu(p-\pi)+\frac{p}{2}-1]}\sum_{k\in\chi_j}|\beta_{jk}|^{\pi}\Big] \\
&\leq {t_\varepsilon}^{p-\pi}\sum_{-1}^{j_0} 2^{j[(\nu+\frac{1}{2}-\frac{1}{p})\frac{p}{q}(q-\pi)]}\tau_j^{\pi} \\
&\leq C{t_\varepsilon}^{p-\pi}2^{j_0[(\nu+\frac{1}{2}-\frac{1}{p})\frac{p}{q}(q-\pi)]}.
\end{aligned}
$$

This is true since $\nu + \frac{1}{2} - \frac{1}{p}$ is also strictly positive because of our constraints. If we now take $2^{j_0 \frac{p}{q}(\nu+\frac{1}{2}-\frac{1}{p})} \sim {t_\varepsilon}^{-1}$ we get the bound

$$ {t_\varepsilon}^{p-q} $$

which is the rate stated in the theorem for this case.

Again, for II, we have, taking now $z_2 = \widetilde{q} > q \ (> \pi)$,

$$ \begin{aligned} \mathrm{II} &\le {t_\varepsilon}^{p-\widetilde{q}} \Big[\sum_{j=j_0+1}^{J} 2^{j[\nu(p-\widetilde{q})+\frac{p}{2}-1]} \sum_{k\in\chi_j} |\beta_{jk}|^{\widetilde{q}} \Big] \\ &\le C{t_\varepsilon}^{p-\widetilde{q}} \sum_{j=j_0+1} 2^{j[(\nu+\frac{1}{2}-\frac{1}{p})\frac{p}{q}(q-\widetilde{q})]} z_j^{\frac{\widetilde{q}}{\pi}} \\ &\le C{t_\varepsilon}^{p-\widetilde{q}} 2^{j_0[(\nu+\frac{1}{2}-\frac{1}{p})\frac{p}{q}(q-\widetilde{q})]} \\ &\le C{t_\varepsilon}^{p-q}. \end{aligned} $$

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