Conformal restriction properties

Wendelin Werner*

Abstract. We give an introduction to some aspects of recent results concerning conformally invariant measures. We focus in this note on the conformal restriction properties of some measures on curves and loops in the plane, and see that these properties in fact almost characterize the measures and allow to classify them. For example, there basically exists a unique measure μ on the set of self-avoiding loops in the plane, such that for any two conformally equivalent domains D and D', the restrictions of μ to the set of loops remaining in D and in D' are conformally equivalent.

This enables to show that a priori different discrete models define the same curves in the scaling limit and exhibit some surprising symmetries. It gives also a way to tie links between these concrete measures on curves and conformal field theory. Important roles in this theory are played by Brownian loops and by the Schramm–Loewner Evolutions (SLE).

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1. A very brief introduction

The last years have seen progress in the mathematical understanding of random two-dimensional structures arising as scaling limits of two-dimensional systems from statistical physics. These probabilistic questions are related to complex analysis considerations (because conformal invariance plays an important role in the description of these objects) and to conformal field theory (that had been developed by theoretical physicists precisely to understand these questions).

Mathematically speaking, one can broadly distinguish two types of questions: Firstly, proving the convergence of the natural discrete lattice-based models from statistical physics to conformally invariant scaling limits. This aspect based on specific lattice models will be discussed in Schramm's and Smirnov's papers in the present proceedings, and will not be the main focus of the present paper. The second type of questions is to define directly the possible continuous limiting objects and to study their

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properties. Two ideas have emerged and can be fruitfully combined to study these continuous objects: The Schramm–Loewner Evolutions (SLE) are random planar curves that are explicitly defined via iterations of random conformal maps, and they appear to be the only ones that combine conformal invariance with a certain Markov property. This shows that they are the only possible conformally invariant scaling limits of interfaces of the critical lattice models. Another instrumental idea is to study how close or how different the random objects defined in different but close domains are, and to see what the conformally invariant possibilities are. This very last approach will be the main focus of the present survey. We warn the reader that we will here remain on a rather general introductory level.

2. Conformal invariance of planar Brownian paths

In this section, we first recall Paul Lévy's result on conformal invariance of planar Brownian paths. We then describe some conformally invariant measures on Brownian loops and Brownian excursions.

2.1. Paul Lévy's theorem. Consider a simple random walk $(S_n, n \geq 0)$ on the square lattice $\mathbb{Z}^2$ (but in fact any planar lattice with some rotational symmetry would do) started from the origin (i.e. $S_0 = 0$). At each integer time, this random walk moves independently to one of its four neighbors with probability $1/4$. In other words, the probability that the first n steps of S are exactly a given nearest-neighbor path on the lattice is equal to 4^{-n}. It is a simple consequence of the central limit theorem that when $N \to \infty$, the law of $(S_{[2Nt]}/\sqrt{N}, t \geq 0)$ converges in some suitable topology to that of a continuous random two-dimensional path $(B_t, t \geq 0)$ with Gaussian independent increments called planar Brownian motion.

It should be noted that planar Brownian paths have a rather complicated geometry. Even if their Lebesgue measure in the plane is almost surely equal to zero, the Hausdorff dimension of a Brownian path is equal to 2 (this can be related to the $\sqrt{N}$ normalization of the simple random walk). Also, there almost surely exists exceptional points of any (including infinite) multiplicity on planar Brownian paths (see [24] and the references therein).

Elementary properties of Gaussian random variables show that the law of the process B is invariant under rotations in the plane, and that it is also scale-invariant (this is also quite clear from the normalization of the random walk) in the following sense: For each given $\lambda > 0$, the laws of $(B_{\lambda^2 t}, t \geq 0)$ and of $(\lambda B_t, t \geq 0)$ are identical. In other words, if one looks at the path of a Brownian motion with a magnifying glass, one sees exactly a Brownian motion, but running at a faster "speed". Paul Lévy (see e.g. [25]) has observed more than fifty years ago that planar Brownian paths exhibit conformal invariance properties that generalize scale-invariance and rotation-invariance, and that we are now describing:

Consider two given conformally equivalent planar domains D and D': These are two open subsets of $\mathbb{C}$ such that there exists an angle-preserving (and orientation-preserving) bijection (i.e. a conformal map) Φ from D onto D'. Recall that when D and D' are two simply connected proper open subsets of the plane, then by Riemann's mapping Theorem, there exists a three-dimensional family of such conformal maps from D onto D'. Consider a point z in D and define its image $z' = \Phi(z)$. Then, define a planar Brownian motion $(B_t, t \geq 0)$ that is started from $B_0 = z$, and denote by T its exit time from the domain D (i.e. $T = \inf\{t \geq 0 : B_t \notin D\}$). For each $t < T$, one can therefore define $\Phi(B_t)$, and when $t \to T$, $\Phi(B_t)$ hits the boundary of $D' = \Phi(D)$. Then:

Theorem 2.1 (Paul Lévy). *The path $(\Phi(B_t), t \leq T)$ is a time-changed planar Brownian motion in D', started at z' and stopped at its first exit time of D'.*

The time-change means that there exists a (random continuous increasing) time-reparametrization $t = t(s)$ such that $(\Phi(B_{t(s)}), s \geq 0)$ is exactly a Brownian motion in D'. In order to state exact conformal invariance properties, we will from now on consider paths defined "modulo increasing time reparametrization".

Lévy's Theorem is nowadays usually viewed as a standard application of stochastic calculus (Itô's formula). It has led to probabilistic approaches to aspects of potential theory and complex analysis.

2.2. Brownian excursions, Brownian loops. It might be desirable to define conformally invariant random objects in a domain $D \subset \mathbb{C}$, but where no marked point in D is given. In Lévy's Theorem, the starting point of the Brownian path is such a special prescribed point. There are (at least) two natural ways to get rid of it without loosing conformal invariance, that both give rise to infinite measures (i.e. measures with an infinite total mass) on Brownian curves.

- A first possibility, described in [22], is to consider Brownian paths that start and end at the boundary of D: Call an excursion in D a continuous path $(e(t), 0 \leq t \leq T)$ such that $e(0, T) \subset D$ and $e(0) \in \partial D$, $e(T) \in \partial D$. Then, for each D, one can define an infinite measure exc_D on the set of "Brownian" excursions in D (with unprescribed time-length) in such a way that the image under a conformal map Φ from D onto D' of the measure exc_D is identical to $\text{exc}_{D'}$ modulo time-change.

  One way to describe the measure in the case where D is equal to the unit disc $\mathbb{U}$ (and therefore in the case of all other simply connected domains D via conformal invariance) is to take the limit when ε goes to zero of ε^{-1} times the law of a Brownian motion started uniformly on the circle $(1-\varepsilon)\partial\mathbb{U}$ and stopped at its first hitting time of the unit circle $\partial\mathbb{U}$. One can also view these measures exc_D as the scaling limits (when $\delta \to 0$) of the measures on discrete excursions on approximations of D by a subset of $\delta\mathbb{Z}^2$ that assign a mass 4^{-n} to each discrete excursion with n steps (see e.g. [12] for precise estimates).

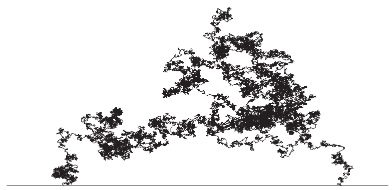

Figure 1. A Brownian excursion in the upper half-plane.

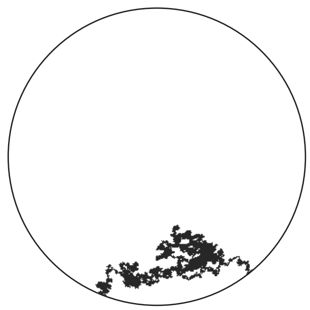

Figure 2. Its conformal image in the unit disc.

- A second possibility, described in [23], that will be important in the present paper, is to consider loops instead of open-ended paths. We say that a continuous planar path $(\ell_t, 0 \le t \le T)$ is a rooted loop if $\ell_0 = \ell_T$. The term rooted is used to emphasize that with this definition, there is a marked point on the loop, namely the starting point ℓ_0. Note that it is possible to re-root a given loop by defining $(\ell'_t = \ell_{t+t_0}, 0 \le t \le T)$ for a given fixed t_0 (where ℓ is extended into a T-periodic function). We may want to say however that ℓ and ℓ' define in fact the same unrooted loop. Hence, we call an unrooted loop the equivalence class of a rooted loop modulo the equivalence defined by this re-rooting procedure. In order to simplify the conformal invariance statements, we will also say that an unrooted loop is defined modulo increasing continuous time-reparametrizations.

  Then [23], there exists a measure M on the set of unrooted (Brownian) loops in the plane with strong conformal invariance properties: For any two conformally equivalent open domains D and $D' = \Phi(D)$, if M_D (resp. M'_D) denotes the measure M restricted to the set of loops that stay in D (resp. D'), then the image measure of M_D under the conformal map Φ from D onto D' is exactly the measure $M_{D'}$.

  One can view this measure M as the limit when δ goes to zero of the measures on discrete unrooted loops in $\delta\mathbb{Z}^2$ that assign a mass 4^{-n} to each loop with n

Figure 3. A Brownian loop.

steps (see e.g. [20] for precise estimates). A direct construction of M goes as follows ([23]): It is easy to define the law $P_{z,T}$ of a Brownian loop with a given time-length T that starts and ends at a given point z. This can be viewed as the conditioning of a Brownian path $(B_t, t \leq T)$ started from $B_0 = z$ by the event $B_T = z$ (this event has zero probability but it is no big deal to make sense of this). Then, one can define a measure $\tilde{M}$ on rooted Brownian loops by integrating the starting point z with respect to the Lebesgue measure in the plane, and the time-length by the measure dT/T^2. Then, M is just the measure on the set of unrooted loops induced by $\tilde{M}$.

Note that exc_D and M are infinite measures (this follows readily from the scale invariance of M and from the scale-invariance of the excursion measure $\text{exc}_{\mathbb{H}}$ in the upper half-plane), so that we can in both cases choose a normalization constant as we wish (i.e. multiply the measures by a well-chosen constant). In fact, the different descriptions of the measures that we did (and will) give differ by a multiplicative constant, and we will not really care here about the exact choice of the constant in the definition of M.

Since these are measures on Brownian paths, they are supported on the set of paths with Hausdorff dimension equal to two, but that the mass of the set of paths that go through any given prescribed point z is equal to zero.

In a way, both these measures are invariant under a larger class of conformal transformations than the killed Brownian motions defined in the previous subsection because no marked starting point is prescribed. Just as killed Brownian motions describe conformally invariant quantities associated to a given marked point such as the harmonic measure, these two measures define also natural conformally invariant quantities that can be related to extremal distances or Schwarzian derivatives for instance.

Let us finally define a further useful Brownian measure, the Brownian excursion measure with prescribed endpoints: The excursion measure $\mathrm{exc}_{\mathbb{U}}$ can be decomposed according to the starting and endpoints of the Brownian excursions. This gives rise for each $A \neq B$ on $\partial\mathbb{U}$ to a probability measure $e_{\mathbb{U},A,B}$ on the set of Brownian excursions from A to B in $\mathbb{U}$. This defines (not surprisingly) again a conformally invariant family of probability measures $(e_{D,A,B})$ where (D, A, B) spans the set $\mathcal{T}$ the set of triplets (D, A, B) such that D is a simply connected proper subset of $\mathbb{C}$ and A and B denote two distinct prime ends of D. When the boundary of D is a smooth self-avoiding loop, this means that A and B are two distinct boundary points. When Φ is a conformal map from D onto D', then "$\Phi(A)$" and "$\Phi(B)$" are then by definition distinct prime ends of $D' = \Phi(D)$.

3. Conformal restriction

We have so far defined some measures on Brownian paths with conformal invariance properties. This means that for each (simply connected) domain, we had a measure m_D on paths in D, and that the family (m_D) is conformally invariant (i.e. $\Phi \circ m_D = m_{\Phi(D)}$). But when $D' \subset D$, it is also natural to compare $m_{D'}$ with the measure m_D restricted to those paths that stay in D'. The conformal restriction property basically requires that these two measures coincide (and that conformal invariance also holds).

3.1. Loops. Suppose that ν is a measure on loops in the plane. As in the rest of the paper, the loops are unrooted and defined modulo increasing time-reparametrizations. For each open domain D, we define ν_D to be the measure ν restricted to the set of loops that stay in D.

Definition 3.1. We say that ν satisfies conformal restriction (resp. conformal restriction for simply connected domains) if for any open domain (resp. open simply connected domain) D and any conformal map $\Phi \colon D \to \Phi(D)$, one has $\Phi \circ \nu_D = \nu_{\Phi(D)}$.

We have already seen one measure satisfying conformal restriction in the previous section: The measure M on Brownian loops in the plane.

Let us now describe a simple argument that shows that all measures that satisfy conformal restriction are closely related. Before that, let us introduce the notion of the filling of a loop. If γ is a loop in the plane, we define its filling $K(\gamma)$ to be the complement of the unbounded connected component of $\mathbb{C} \setminus \gamma$. In other words, $K(\gamma)$ is obtained by filling in all the bounded connected components of the complement of γ. Clearly, any measure on loops defines a measure on their fillings, and we can also define the conformal restriction property for measures on fillings.

Proposition 3.2 ([42]). *Up to multiplication by a positive constant, there exists a unique measure on fillings that satisfies conformal restriction for simply connected domains. It can be defined as the measure on filling of Brownian loops.*

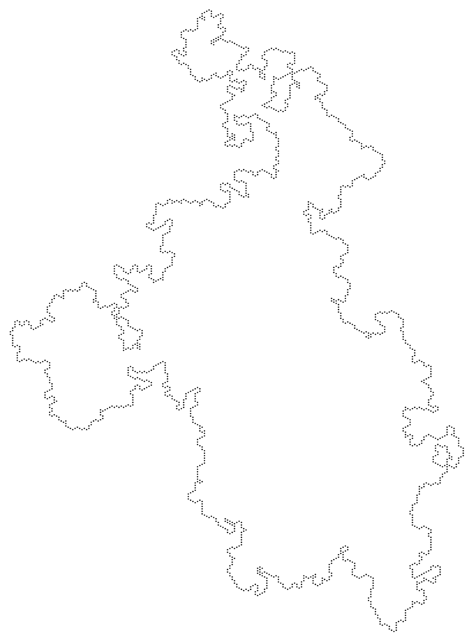

Figure 4. A self-avoiding loop.

Proof (*sketch*). In this proof, we will always discuss conformal restriction *for simply connected domains*. The existence part of the proposition follows from the fact that the measure M on Brownian loops exists and satisfies conformal restriction (so that the fillings of Brownian loops satisfy conformal restriction as well). It remains to prove the uniqueness statement.

Consider the family $\mathcal{U}$ of conformal maps φ from some (unprescribed) simply connected subset U of the unit disc $\mathbb{U}$ containing the origin onto the unit disc $\mathbb{U}$, such that $\varphi(0) = 0$ and $\varphi'(0)$ is a positive real number. Riemann's mapping theorem shows that for any simply connected domain $U \subset \mathbb{U}$ with $0 \in U$, there exists a unique $\varphi = \varphi_U \in \mathcal{U}$ from U onto $\mathbb{U}$. Note that $\mathcal{U}$ is closed under composition: If φ_U and φ_V are in $\mathcal{U}$, then so is $\psi = \varphi_U \circ \varphi_V$ (it is a conformal map from $\varphi_V^{-1} \circ \varphi_U^{-1}(\mathbb{U}) = \varphi_V^{-1}(U)$ onto $\mathbb{U}$ with the right properties at the origin). Note that of course, $\log \psi'(0) = \log \varphi_U'(0) + \log \varphi_V'(0)$. It is also straightforward to check that $\varphi_V'(0) \geq 1$ because $V \subset \mathbb{U}$.

Suppose now that a measure ν on fillings satisfies conformal restriction. Let us define for each $\varphi_U \in \mathcal{U}$,

$$A(\varphi_U) = \nu(\{K : 0 \in K,\ K \subset \mathbb{U},\ K \not\subset U\}).$$

This is the mass of fillings containing the origin, that stay in $\mathbb{U}$ but not in U. Then, it is easy to see that

$$A(\varphi_U \circ \varphi_V) = A(\varphi_U) + A(\varphi_V).$$

Indeed, there are two types of fillings that contain the origin, stay in $\mathbb{U}$ but not in $\varphi_V^{-1} \circ \varphi_U^{-1}(\mathbb{U})$:

- Those that do not stay in $V = \varphi_V^{-1}(\mathbb{U})$ and the set of these fillings has a ν-mass equal to $A(\varphi_V)$ by definition.
- Those that stay in $V = \varphi_V^{-1}(\mathbb{U})$ but not in $\varphi_V^{-1}(\varphi_U^{-1})(\mathbb{U}) = \varphi_V^{-1}(U)$. But by conformal invariance (via the mapping φ_V), this set is conformally equivalent to the set of loops that stay in $\mathbb{U}$ and not in U. So, its ν-mass is $A(\varphi_U)$.

Rather soft considerations (for instance involving Loewner's approximation of any mapping in $\mathcal{U}$ by iterations of slit mappings) then imply that the functional A is necessarily of the form $A(\varphi_U) = c \log \varphi_U'(0)$ for a positive constant c.

Hence, it follows by conformal invariance that for each $z \in D' \subset D$, the ν-mass of the set of fillings that contain z, stay in the simply connected domain D but not in the simply connected domain D' is equal to c times the logarithm of the derivative at z of the conformal map from D' onto D that fixes z and has positive derivative at z. Soft arguments (of the type "a finite measure is characterized by its values on a intersection-stable set that generates the σ-field") then show that (for each choice of c) this characterizes the measure ν uniquely. This implies the uniqueness part of the proposition. □

It is possible to show that the boundary of a Brownian loop is almost surely a self-avoiding loop (the fact that it is a continuous loop is straightforward, but the fact that it has no double point requires some estimates, see e.g. [4]). Hence, the proposition shows that modulo multiplication by a positive constant, there is a unique measure μ on self-avoiding loops that satisfies conformal restriction for simply connected domains. As we shall see later, it turns out that it satisfies also the general conformal restriction property.

In [15], [16] (see also Schramm's contribution in these proceedings), it is proved that the Hausdorff dimension of the outer boundary of a Brownian path is almost surely $4/3$ (the proof uses SLE considerations and we shall explain why later in this paper). Hence:

Corollary 3.3. *For the* (*up-to-constants*) *unique measure on fillings that satisfies conformal restriction, the boundary of the filling is almost surely a self-avoiding loop with dimension* $4/3$.

3.2. The chordal case. Suppose that for each $(D, A, B) \in \mathcal{T}$, we have the law $P_{D,A,B}$ of a random excursion from A to B in D. We say that the family $(P_{D,A,B})$ is conformally invariant if for any D, A, B and any conformal map from D onto some domain $D' = \Phi(D)$, the image measure of $P_{D,A,B}$ under Φ is the measure $P_{\Phi(D),\Phi(A),\Phi(B)}$.

This implies in particular that $P_{D,A,B}$ is invariant under any conformal map from D onto itself that preserves the boundary points A and B. For instance, for $D = \mathbb{H}$, $A = 0$ and $B = \infty$, this means that $P_{\mathbb{H},0,\infty}$ is scale-invariant (i.e. for each $\lambda > 0$, γ and $\lambda\gamma$ have the same law modulo time-reparametrization). We then say that the probability measure $P_{D,A,B}$ is conformally invariant.

Conversely, if one has a probability measure P on excursions from A_0 to B_0 in D_0 for some given triplet (D_0, A_0, B_0) that is conformally invariant, one can simply define for each D, A, B in $\mathcal{T}$ the measure $P_{D,A,B}$ to be the conformal image of P under a conformal map from (D_0, A_0, B_0) onto (D, A, B). The obtained family $(P_{D,A,B})$ is then conformally invariant.

We say that the family $(P_{D,A,B})$ is restriction-invariant if for any D, A, B, and any simply connected subset D' of D such that the distance between $\{A, B\}$ and $D \setminus D'$ is positive (this implies in particular that A and B are on $\partial D'$), one has

$$P_{D,A,B}(\cdot \mid \gamma \subset D') = P_{D',A,B}(\cdot).$$

In other words, if γ is defined under $P_{D,A,B}$, the conditional law of γ given $\gamma \subset D'$ is exactly $P_{D',A,B}$.

Definition 3.4. We say that the probability measure $P_{D,A,B}$ for some $(D, A, B) \in \mathcal{T}$ satisfies conformal restriction if:

- It is conformally invariant.
- The conformally invariant family that it defines is restriction-invariant

Note that an excursion γ from A to B in D defines also a filling $K(\gamma)$, and that one can generalize the conformal restriction property to fillings also.

For a fixed triplet D, A, B, we call $\mathcal{D}_{D,A,B}$ the set of all simply connected domains $D' \subset D$ such that the distance between $D \setminus D'$ and $\{A, B\}$ is strictly positive. For each such D', we define a conformal map from D' back onto D with $\Phi(A) = A$ and $\Phi(B) = B$. In the case where ∂D is smooth in the neighborhood of A and B, one can define $\Phi'(A)$ and $\Phi'(B)$ (which are real numbers) and note that the product of these two derivatives does not depend on which Φ (in the possible one-dimensional family of maps) one did choose. When ∂D is not smooth in the neighborhood of A and B, it is still possible to make sense of the quantity "$\Phi'(A)\Phi'(B)$" by conformal invariance (map D onto the unit disc, and look at the corresponding quantity for the image of A, B and D'). In short, the quantity $\Phi'(A)\Phi'(B)$ is a conformally invariant quantity that measures how smaller D' is compared to D, seen from the two points/prime ends A and B.

Theorem 3.5 ([17]). *For each triple* $(D, A, B) \in \mathcal{T}$*, there exists exactly* (*and in particular: no more than*) *a one-parameter family of measures on fillings that satisfy conformal restriction. It is parametrized by a number* $\alpha \in [5/8, \infty)$ *and for each* α*, the corresponding measure* $P^{\alpha}_{D,A,B}$ *is characterized by the property that for each* $D' \in \mathcal{D}_{D,A,B}$*,*

$$P^{\alpha}_{D,A,B}(K \subset D') = (\Phi'(A)\Phi'(B))^{\alpha}.$$

Proof (*sketch*). The uniqueness part is analogous to the loop case: By conformal invariance, we may choose D, A, B to be $\mathbb{U}$, -1, 1. Then, the set $\mathcal{D} := \mathcal{D}_{\mathbb{U},-1,1}$ is the family of simply connected subsets U of $\mathbb{U}$ such that $\mathbb{U} \setminus U$ is at positive distance from 1 and -1. For each such U, we define $\psi = \psi_U$ to be the unique conformal map from U onto $\mathbb{U}$ such that $\psi(-1) = -1$, $\psi(1) = 1$ and $\psi'(-1) = 1$. The family of these conformal maps is closed under composition, and for two such maps ψ_1 and ψ_2, $(\psi_1 \circ \psi_2)'(1) = \psi_1'(1)\psi_2'(1)$.

Suppose that the measure P on fillings of excursions from -1 to 1 in $\mathbb{U}$ satisfies conformal restriction. We then define for each such $U \in \mathcal{D}$,

$$A(\psi_U) = P(K \subset U).$$

Conformal restriction implies readily that $A(\psi_U \circ \psi_V) = A(\psi_U) \times A(\psi_V)$ for all U and V in $\mathcal{D}$, and this leads to the fact that there exists a positive constant α such that

$$P(K \subset U) = A(\psi_U) = \psi_U'(1)^{\alpha}.$$

But the probability measure P is fully characterized by the knowledge of all the probabilities $P(K \subset U)$ for $U \in \mathcal{D}$.

It then remains to see that for each $\alpha \geq 5/8$, these identities indeed describe a probability measure on fillings, and that when $\alpha < 5/8$, no such measure exists. The way we prove this in [17] is that we explicitly construct the measure when $\alpha \geq 5/8$ using the Schramm–Loewner Evolution (SLE) process. For $\alpha < 5/8$, we also construct what would be the unique possible candidate (that satisfies a weaker condition – called the one-sided conformal restriction property – than the conformal restriction property that we described) for P (via SLE or Brownian means), and we show that this candidate fails to satisfy the actual conformal restriction property. □

It is easy to check that the Brownian excursions from A to B in D (and their fillings therefore also) defined by $e_{D,A,B}$ do satisfy conformal restriction for $\alpha = 1$, so that for $P^1_{D,A,B}$ the boundary of the filling is almost surely supported on sets of Hausdorff dimension 4/3.

Let us give a partial description of the boundary of these fillings for general α in terms of Brownian excursions. Let us stick to case of the triplet $\mathbb{U}$, -1, 1. Suppose that K is a filling satisfying conformal restriction. Then it turns out that $K \cap \partial\mathbb{U} = \{-1, 1\}$ and that the complement of K in $\mathbb{U}$ consists of two connected components: The upper one O^+ such that ∂O^+ contains the upper half-circle $\partial_+ := \{e^{i\theta},\ \theta \in (0, \pi)\}$ and the lower one O^-, such that ∂O^- contains the lower semi-circle ∂_-. The boundary

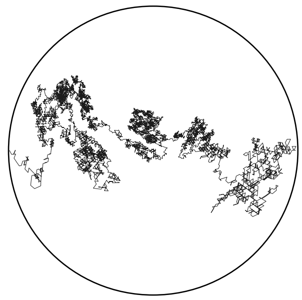

Figure 5. A Brownian excursion from -1 to 1 in the unit disc (sketch).

of O^+ (resp. O^-) then consists of the upper (resp. lower) semi-circle and a continuous curve γ^+ (resp. γ^-) joining -1 to $+1$ in $\mathbb{U}$. It is then not difficult to see that the law of γ^+ is characterized by the fact that for any $U \in \mathcal{D}$, such that $\mathbb{U} \setminus U$ is at positive distance of the lower semi-circle (i.e. $\mathbb{U} \setminus U$ is attached to the upper semi-circle)

$$P^\alpha(\gamma^+ \subset U) = \varphi'_U(1)^\alpha$$

(we will call $\mathcal{D}^+$ this subset of $\mathcal{D}$). One way to construct such a random curve uses a Poissonization argument and the Brownian excursion measure that we described earlier. Since a similar Poissonization argument will be useful in another setup a little bit later, let us briefly describe this classical idea in abstract terms:

Suppose that N is a σ-finite measure without atoms on some space $\mathcal{X}$. We can define the law of a random countable family $X = \{X_j, j \in J\}$ of elements of X in such a way that:

- For each $A_1, A_2 \subset \mathcal{X}$ in the σ-field on which N is defined, such that $A_1 \cap A_2 = \emptyset$, the random families $X \cap A_1$ and $X \cap A_2$ are independent.
- For each A_1 as above, the probability that $X \cap A_1$ is empty equals $\exp(-N(A_1))$.

The law of X is in fact characterized by these two properties. It is easy to see that for each A, the cardinality of $X \cap A$ is a Poisson random variable with mean $N(A)$ (so that it is a.s. infinite if and only if $N(A) = \infty$). X is called a Poisson point process with intensity N.

Note that if X_1 and X_2 are two independent Poisson point processes on the same space $\mathcal{X}$ with respective intensity N_1 and N_2, then $X_1 \cup X_2$ is a Poisson point process with intensity $N_1 + N_2$.

Using this idea, one can define on the same probability space a collection $(X_c, c \geq 0)$ of Poisson point processes in such a way that $X_c \subset X_{c'}$ for all $c \leq c'$, and such that the intensity of X_c is cN. One intuitive way to view this is to say that with time, elements of $\mathcal{X}$ appear independently. During a time-interval dt, an element of a set $A \subset \mathcal{X}$ will appear with probability $dt \times N(A)$. Then, X_c denotes the family of elements that did appear before time c.

Let us now use this construction for a measure N on the space of excursions in $\mathbb{U}$. More precisely, we define $\mathcal{X}$ the set of excursions in $\mathbb{U}$ that start and end on the lower semi-circle ∂_-, and we define N to be $\text{exc}_{\mathbb{U}}$ restricted to this set of excursions.

Hence, for each c, the previous procedure defines a random countable collection of Brownian excursions $E_c = (e_j, j \in J_c)$ starting and ending on the negative half-circle. Despite the fact that this collection is almost surely infinite (because the total mass of N is infinite), the total number of excursions of diameter greater than ε is almost surely finite for all positive ε (because the N-mass of this set of excursions is finite). In particular, this implies that the "upper boundary" γ^+ of the union of all excursions in E_c does not intersect the upper semi-circle ∂_+. It does not exit a given $U \in \mathcal{D}^+$ if and only if no excursion in E_c does exit U, and by definition, this happens with probability $\exp(-cN(\{\gamma : \gamma \not\subset U\}))$.

The conformal restriction property of the excursion measure shows that for each $U \in \mathcal{D}^+$, the image under φ_U of the measure N restricted to the set of excursions that stay in U is exactly equal to N. It follows readily from this fact that $\exp(-N(\{\gamma : \gamma \not\subset U\})) = \varphi_U'(1)^{\alpha_1}$ for some α_1. Hence, for each $\alpha > 0$, if one chooses $c = \alpha/\alpha_1$, the curve γ^+ does indeed satisfy $P(\gamma^+ \subset U) = \varphi_U'(1)^{c\alpha_1} = \varphi_U'(1)^{\alpha}$.

The fact that $\alpha < 5/8$ is not possible corresponds to the fact that the probability that γ^+ goes "below" the origin becomes larger than $1/2$, which is not possible for symmetry reasons if it is equal to the upper boundary of a filling satisfying conformal restriction.

For more precise statements and also other possible descriptions of the joint law of (γ^+, γ^-), see [17], [39], [40].

4. Related models

So far, we have defined only measures on Brownian curves, and we have basically shown that any measure satisfying conformal restriction defines the same outer boundary as that of these Brownian measures. The theory becomes interesting when we note that some a priori different measures do also satisfy conformal restriction.

4.1. Percolation. We now very briefly describe the percolation model that has been proved by Smirnov [35] to be conformally invariant in the scaling limit. Consider the honeycomb lattice (the regular tiling of the plane by hexagons) with mesh size δ. Each hexagon is colored independently in black or in white with probability $1/2$. Then, we are interested in the connectivity properties of the set of white (resp. black) cells. We call white (resp. black) cluster a connected component of the union of the white (resp. black) cells. This model is sometimes called "critical site-percolation on the triangular lattice".

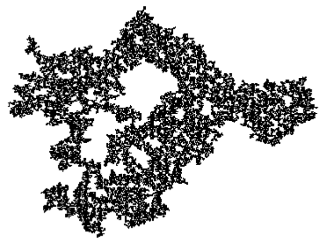

Figure 6. A rescaled large percolation cluster.

By now classical arguments due to Russo, Seymour and Welsh show that the number of clusters that are of diameter $\varepsilon > 0$ in the unit disc remains tight when $\delta \to 0$. This suggests the existence of a scaling limit for the joint law on all clusters when δ (in an appropriately chosen topology). Smirnov [35] proved the existence of the limit of certain observables (the crossing probabilities) and their conformal invariance.

A consequence of this result is [36] that it is possible to use SLE computations from [14], [15] and earlier results from Kesten [11] to deduce the existence and the value of the critical exponents for critical percolation as predicted by theoretical physicists such as Cardy, Duplantier, Saleur (see e.g. the references in [36]). But, we would here like to focus on the conformal restriction aspect of the scaling limit of percolation and its consequences. We will remain on a heuristic level, but what follows can be made rigorous:

A percolation configuration is described by its white (say) clusters $(C_j,\ j \in J)$. Smirnov's result can be shown to imply (see [5]) the convergence in law of this family when $\delta \to 0$ to the joint law of a collection of "clusters" $(C_j,\ j \in J)$ in the plane. A slightly weaker statement is that the measure on clusters π^δ that assigns to each possible cluster the probability that this cluster indeed occurs converges when $\delta \to 0$ towards a measure on "clusters" π. The measure π satisfies conformal restriction. This is due to the combination of conformal invariance (due to Smirnov's result) and of the independence properties of percolation from which restriction immediately follows in the scaling limit. Hence:

Proposition 4.1. *The measure π on scaling limits of critical percolation clusters satisfies conformal restriction.*

So, π defines exactly the same fillings as (a multiple) of the Brownian loop measure M, and it defines a measure on outer boundaries that is exactly a multiple of μ. In other words, the shape of the outer perimeter of a very large percolation cluster has (in the scaling limit) the same law than the outer boundary of a Brownian loop.

4.2. The self-avoiding walk conjectures. A classical open problem is to understand the behavior of very long self-avoiding paths, sampled uniformly among all such long self-avoiding paths on some planar lattice with a given starting point and a given length N, in the limit when $N \to \infty$.

It is believed that in the scaling limit (for regular periodic lattices with some rotational symmetry) these paths exhibit conformal invariance properties. This led to various striking predictions by theoretical physicists concerning this model and its critical exponents.

For instance, it is believed that the diameter of a typical self-avoiding path with N steps is of the order of $N^{3/4}$. This can be loosely phrased in terms of "fractal dimension" since it means that one requires N steps of size $N^{-3/4}$ to cover a long self-avoiding walk of macroscopic size on the lattice $N^{-3/4}\mathbb{Z}^2$. More precisely, this could mean that in the scaling limit, self-avoiding walks converge in law to some continuous measure on paths supported on the set of paths with dimension equal to $4/3$.

Note that the number of self-avoiding walks of length N on $\mathbb{Z}^2$ that start at the origin can easily (via a sub-multiplicativity argument) be shown to behave like $\lambda^{N+o(N)}$ when $N \to \infty$, where λ is a positive real number called the connectivity constant of $\mathbb{Z}^2$. One of the striking conjectures in this field is the more precise prediction $\lambda^N N^{11/32+o(1)}$ by Nienhuis [28].

Here are two possible ways to state this existence of scaling-limit conjecture (in the case of the square lattice):

- Self-avoiding loops: The measure on self-avoiding loops on $\delta\mathbb{Z}^2$ that assigns a mass λ^{-N} to each loop with N steps has a (non-trivial) limit when $\delta \to 0$.
- Excursions: The probability measure on self-avoiding excursions from -1 to 1 in an approximation of $\mathbb{U}$ by a sublattice of $\delta\mathbb{Z}^2$ that assigns a probability proportional to λ^{-N} to each excursion with N steps converges (when $\delta \to 0$) to a (non-trivial) scaling limit.

In the first case, the scaling limit is then a measure S supported on the set of loops in the plane. In the second one, it is then a probability measure P^S on the set of excursions from -1 to 1 in $\mathbb{U}$.

If one assumes furthermore that these measures exhibit conformal invariance properties, then S should be a measure on self-avoiding loops satisfying conformal restriction: By the previously described results, it is therefore a multiple of the measure μ on outer boundaries of Brownian loops and of the measure on outer boundaries of percolation clusters. Similarly, we get that P^S should satisfy chordal conformal restriction. Hence, it should be a measure on excursions without double points that coincides with one of the P^α's. This gives an explanation (but not a proof) of the $4/3$-dimension conjecture for self-avoiding walks.

Let us note that in his book [26], Mandelbrot had already proposed the name "self-avoiding Brownian motion" for the outer boundary of a planar Brownian loop.

The above results show that this would be indeed an appropriate name.

5. Related SLEs

The (chordal) Schramm–Loewner Evolutions (SLE) first introduced in [31] are conformally invariant random planar excursions in a domain with prescribed endpoints. They are defined via iterations of random conformal maps and they are the only ones satisfying a certain Markov property. Since the discrete analogue of this Markovian property is obviously satisfied by the interfaces of many discrete lattice-models from statistical physics (including for instance percolation), this shows that if these discrete interfaces converge to conformally invariant scaling limits, then they have to be one of the SLE curves. For details on the definition and properties of SLEs, their relations (conjectured and proved) to lattice-models, there are now many surveys, lecture notes, a book (e.g. [13], [37] and the references therein); see also Schramm's contribution to the present ICM proceedings.

There exists a one-parameter family of SLE's: For each $\kappa > 0$, the SLE with parameter κ (in short: SLE_κ) is a mathematically well-defined random planar excursion joining prescribed boundary points in a simply connected domain [30], [18]. One can then see if these random excursions satisfy conformal restriction (in the chordal case). It turns out that:

Proposition 5.1 ([17]). *$SLE_{8/3}$ is a random excursion without double points that satisfies chordal restriction. Its law is exactly $P^{5/8}$. No other SLE satisfies chordal conformal restriction.*

In fact, one can prove that it is the only measure supported on excursions without double points that satisfies chordal conformal restriction (i.e. that for all $\alpha > 5/8$, the measure P^α is not supported on self-avoiding curves). Hence, the $\mathrm{SLE}_{8/3}$ is the conjectural scaling limit of self-avoiding excursions, i.e. $P^S = P^{5/8}$. Not surprisingly given all what we have said so far, it can be proved directly that it is supported on the set of excursions with Hausdorff dimension $4/3$ [17], [2]. The computation of the critical exponents for SLE (e.g. [14], [15]) allow also to recover the physicists' predictions on critical exponents such as the $11/32$ mentioned above (see e.g. [19]).

Also, there is a rather direct relation between discrete self-avoiding loops and self-avoiding excursions (the self-avoiding excursion tells how to finish a loop if we know part of it). This suggests a direct relation between the outer boundaries of planar Brownian loops and the $\mathrm{SLE}_{8/3}$ processes. Indeed (see e.g. [23]), it is possible to define a measure on $\mathrm{SLE}_{8/3}$ loops and to see that it is a measure on self-avoiding loops in the plane that satisfies conformal restriction:

Proposition 5.2. *The measure μ can be viewed as a measure on $SLE_{8/3}$ loops.*

In fact, this has a deeper consequence, which is not really surprising if one thinks of μ in terms of the conjectural scaling limit S of the measure on discrete self-avoiding loops:

Theorem 5.3 ([42]). *The measure μ on self-avoiding loops satisfies conformal restriction also for non-simply connected domains D.*

A particular instance of the theorem is that the measure μ is invariant under the inversion $z \mapsto 1/z$. This implies [42] that the inner boundaries of Brownian loops (and those of the scaling limits of critical percolation clusters) have exactly the same distribution than the outer boundaries. More precisely, if one looks at the boundary of the connected component that contains the origin of the complement of a Brownian loop (defined under M) then it is defined under exactly the same measure as the outer boundary. This is by no means an obvious fact.

Another consequence is the following:

Corollary 5.4 ([42]). *It is possible to extend the definition of the planar measure μ on self-avoiding loops to any Riemann surface* (*possibly with boundaries*) *in such a way that conformal restriction still holds.*

This gives a direct description of various conformally invariant quantities in the framework of Riemann surfaces.

The SLE_6 process can be shown (see e.g. [14], [15]) to be the only SLE satisfying a so-called locality property that makes it the only possible candidate for the (conformally invariant) scaling limit of percolation interfaces. In fact, using Smirnov's result [35] and ideas, it is possible to deduce [5] that SLE_6 is indeed this scaling limit for critical percolation on the triangular lattice. Hence, it should not be surprising that it is possible to define directly (from the definition of SLE_6) conformally invariant measures on loops and excursions that satisfy conformal restriction (see e.g. [17]). This is one of the ways to see that chordal restriction for $\alpha = 2$ is very closely related to the loop measure μ.

6. Restriction defect

Most models arising from statistical physics should however not satisfy conformal restriction in the scaling limit. Self-avoiding walks and percolation are in this respect rather exceptional cases. We now describe how one can extend the conformal restriction property to cover the more generic cases. It is useful to start with a specific model to illustrate the basic ideas and to show why the Brownian loop-soup can be useful.

6.1. Loop-erased random walks. Suppose that $S = (S_n, n \le N)$ is a discrete nearest neighbor-walk of length N on a finite connected graph G. It is as a path joining the two points $o = S_0$ and $e = S_N$ that can have double points. One can however associate to S a path from o to e without double-points by following S and erasing the loops as they appear. This gives rise to the loop-erasure $L = L(S)$ of S. It is the only simple path from $o = L_0$ to $e = L_p$ (the length p of L is not greater than N but it can be smaller and it depends on the length of the loops erased during this procedure) with the property that for each $i \leq p - 1$, $L_{i+1} = S_{n_i+1}$, where $n_i = \sup\{n \leq N : S_n = L_i\}$.

If we are given the two points o and e, we can choose S randomly to be a simple random walk on the graph, started at o and stopped at its first hitting of e. Its loop-erasure $L = L(S)$ is then the so-called loop-erased random walk from o to e. It has many nice combinatorial features, that are not obvious at first sight. For instance, the law of the loop-erased random walk from o to e and of the loop-erased random walk from e to o are the same (modulo time-reversal of course). It can also be interpreted as the law of the unique (simple) path joining o to e in a spanning tree chosen uniformly among all spanning trees of the graph G (i.e. choose uniformly a subgraph of G with just one connected component but no cycle, and look at the unique path joining o to e in this subgraph). This result by Pemantle [29] has been extended by Wilson into a complete construction of a uniformly chosen spanning tree of G using loop-erased random walks [43]. It shows that loop-erased random walks belong to a wider general class of models from statistical physics (the random-cluster models) that includes also the Ising models.

A fine-grid approximation of the Brownian excursion measure $e_{\mathbb{U},-1,1}$ goes as follows: Consider a fine-mesh approximation of the unit disc with two boundary points o and e close to -1 and 1, and consider a simple random walk started from o, stopped at e, and conditioned to exit $\mathbb{U}$ through e.

Theorem 6.1 ([18]). *The loop-erasure of this discrete excursion converges when the mesh-size converges to zero to a conformally invariant scaling limit, the SLE_2 from* -1 *to* 1 *in* $\mathbb{U}$. *Similarly, for any triplet $(D, A, B) \in \mathcal{T}$, the loop-erasure of a fine-grid approximation of an excursion defined under $e_{D,A,B}$ converges to the SLE_2 from A to B in D.*

For a given $U \subset \mathbb{U}$ that still has -1 and 1 on its boundary, it happens with positive probability that the loop-erasure L of the discrete excursion S stays in U, but that the path S does exit U (i.e. one of the erased loops went out of U). This feature pertains in the scaling limit and shows that conformal restriction is not satisfied by SLE_2. The lack of restriction can be quantified in terms of the erased random walk loops (i.e. in the scaling limit in terms of a quantity involving Brownian loops). More precisely, for a given simple nearest neighbor path from $o \sim -1$ to $e \sim 1$ on the $\delta\mathbb{Z}^2$-approximation of $U \subset \mathbb{U}$, the ratio between the probability that $L = l$ for the LERW from o to e in U and the probability that $L = l$ for the LERW in $\mathbb{U}$ is given by

$$F_\delta(l) = \text{cst}(U) P_{\mathbb{U}}(\text{none of the erased loops did exit } U \mid L = l).$$

This function F_δ converges to a non-trivial function F when $\delta \to 0$ that measures the restriction-defect of SLE_2 and that can be expressed in terms of Brownian loops.

6.2. The Brownian loop soup. Consider the (properly normalized) Brownian loop-measure M. Recall that it is a measure on the set of unrooted Brownian loops in the entire plane. For each $c > 0$, we define a Poisson point process with intensity cM. This is a random countable collection $\{b_j, j \in J\}$ of Brownian loops in the plane.

For each domain D, we define $J(D) = \{j \in J : b_j \subset D\}$. It is clear from the definition that this corresponds to a Poisson point process with intensity cM_D.

In [23], we show that:

Proposition 6.2 ([23]). *The function $F(l)$ is equal to the probability that no loop in the loop-soup with intensity $2M_{\mathbb{U}}$ intersects both the excursion l and the complement of U.*

This indicates that the loops that have been erased correspond to the loops in the loop-soup that the path l intersects. This is not so surprising if one thinks of Wilson's algorithm (that in some sense shows that the law of the constructed uniform spanning tree is independent of the erased loops).

It shows [23] that if one adds to an SLE_2 the loops that it intersects in a Brownian loop-soup, one recovers exactly a path satisfying conformal restriction (in fact with parameter $\alpha = 1$, the one of the Brownian excursion $\mathrm{exc}_{\mathbb{U},-1,1}$).

A similar coupling of the SLE_κ's for $\kappa < 8/3$ with a Brownian loop-soup of parameter $c = c(\kappa) = (8 - 3\kappa)(6 - \kappa)/2\kappa$. By adding the loops of this loop-soup to the SLE curve, one compensates its lack of restriction and constructs a filling that satisfies conformal restriction with parameter $\alpha = (6 - \kappa)/2\kappa$. These relations correspond to the relation between the central charge $(-c)$, the highest weight (α) and the degeneracy factor $(\kappa/4)$ of degenerate highest-weight representations of the Virasoro Algebra, as predicted by conformal field theory (see e.g. [9], [1], [3]).

6.3. Loop-soup clusters, CLEs. This does not describe the type of restriction-defects of the SLE's with parameter $\kappa > 8/3$ that should arise as scaling limits of various lattice models, corresponding in the physics language to models with positive central charge. Loosely speaking, these are the curves that are attracted by the boundaries of a domain (as opposed for instance to the SLE_2 that was "repelled" from the boundary). The previous case $\kappa < 8/3$ corresponded to a negative central charge.

For this, it is useful to consider the geometry of the union of all loops in a loop-soup of intensity $c\mu_{\mathbb{U}}$ (recall that the measure $\mu_{\mathbb{U}}$ corresponds to the outer boundaries of the Brownian loops defined by $M_{\mathbb{U}}$). This loop-soup is a countable collection $C_c = \{\ell_j, j \in J_c\}$ of self-avoiding loops in the unit disc that can overlap with each other. Recall that can couple all C_c's in such a way that $c \mapsto C_c$ is increasing.

When c is large and fixed, it is not difficult to see that almost surely every point in $\mathbb{U}$ is surrounded by a loop in C_c, so that all the loops hook up into one single connected component i.e. the set $\bigcup_{j \in J_c} \ell_j$ has just one connected component.

On the other hand, when c is small, it is also easy for instance by coupling this problem with the so-called fractal percolation (sometimes also called Mandelbrot percolation) studied in [7], [27] to see that this phenomenon does not pertain: The

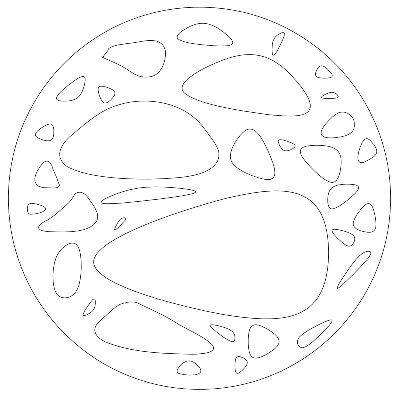

Figure 7. A CLE (very very sketchy).

set $\bigcup_{j \in J_c} \ell_j$ has countably many connected components. The outermost boundaries of these clusters of loops define a family of non-overlapping and non-nested loops $\boldsymbol{u}^c = \{u_i^c,\ i \in I_c\}$ in $\mathbb{U}$.

This leads to the following definition [34]:

Definition 6.3. Suppose that $\boldsymbol{u} = \{u_i, i \in I\}$ is a random collection of non-intersecting and non-nested self-avoiding loops in $\mathbb{U}$. We say that it is a simple conformal loop-ensemble (CLE) if the following properties hold:

- It is invariant under the conformal transformation from $\mathbb{U}$ onto itself. This allows to define the law P_U of the collection of loops in any simply connected domain U by taking the conformal image of $\boldsymbol{u}$.
- Let U be any simply connected subset of $\mathbb{U}$ with $d(\mathbb{U} \setminus U, 1) > 0$. Consider $I' = \{i \in I, u_i \not\subset U\}$ and let $\tilde{U}$ denote the connected component of $U \setminus \bigcup_{i \in I'} u_i$ that has 1 on its boundary. Then, conditionally on $\{u_i, i \in I'\}$, the law of $\{u_i, i \in I$ and $u_i \subset \tilde{U}\}$ is $P_{\tilde{U}}$.

Loosely speaking, this means that each loop (once it is discovered) plays the role of the boundary of the domain in which the others are yet to be discovered. Note that a CLE almost surely is an infinite collection of loops (because the number of loops contained in $\mathbb{U}$ and in $\tilde{U} \subset \mathbb{U}$ have the same law).

The previous considerations show that the outermost boundaries of cluster of loops for sub-critical (i.e. for small c) loop-soups, are conformal loop ensembles (so that CLEs exist). This gives rise to measures on loops that do not satisfy conformal restriction, but have the same type of restriction defect as that of SLEs for $\kappa \in (8/3, 4]$. The intensity c of the loop-soup corresponds to the central charge of the model.

Conformal loop-ensembles (and SLEs) arise also in the context of level-lines (or flow-lines) of the Gaussian Free Field [33] in the ongoing work of Oded Schramm and

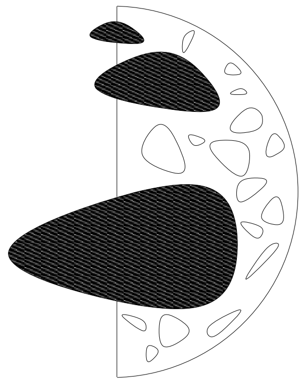

Figure 8. The CLE in $\tilde{U}$ when U is the semi-disc.

Scott Sheffield [32]. Combining all these arguments should [34] describe all CLEs as loop-cluster boundaries and their boundaries as SLE loops for $\kappa \le 4$.

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Université Paris-Sud, Laboratoire de mathématiques, Bât. 425, 91405 Orsay, France
and
Ecole Normale Supérieure, Département de mathématiques et applications, 45, rue d'Ulm, 75230 Paris cedex 05, France

E-mail: wendelin.werner@math.u-psud.fr