Towards a structure theory for matrices and matroids

Jim Geelen, Bert Gerards, and Geoff Whittle*

Abstract. We survey recent work that is aimed at generalizing the results and techniques of the Graph Minors Project of Robertson and Seymour to matrices and matroids.

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1. Introduction

We are currently undertaking a program of research aimed at extending the results and techniques of the Graph Minors Project of Robertson and Seymour to matrices and matroids. Here we report on where we stand and where we expect to go.

In particular, we discuss the structure of "minor-closed" classes of matrices over a fixed finite field. This requires a peculiar synthesis of graphs, topology, connectivity, and algebra. In addition to proving several long-standing conjectures in the area, we expect the structure theory will help to find efficient algorithms for a general class of problems on matrices and graphs.

Most combinatorial computational problems are trivial in the sense that they are typically finite. However, even for modest size problems, enumerating the possibilities is practically infeasible; it often results in algorithms whose running time is exponential in the size of the problem. We seek smarter, more efficient, algorithms. In the theory of algorithms *efficient* typically means that the running time is polynomial in the size of the problem.

Often the problems are modeled by graphs (networks) or matrices. The better picture we have of the model, the more likely it is that we can develop a quick algorithm for the problem. For instance, the problem at hand may be more tractable if the modeling graph can be drawn in the plane, or some other particular surface, without crossings. Then it is relevant that we can test efficiently if the graph has such advantageous appearance. Surface embeddability, and other related properties, are preserved when deleting an edge from the graph or contracting an edge (*contracting* means deleting the edge and identifying its ends). The result of any series of such deletions and contractions is called a *minor* of the graph. In this terminology, testing

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surface embeddability is testing a particular minor-closed graph property. So motivated by real-word computational problems, we end up with the fundamental question if minor-closed graph properties can be tested efficiently.

That this is possible indeed for any fixed minor-closed graph property is one of the consequences of the ground-breaking work of Robertson and Seymour in their Graph Minors Project. One major outcome of this project is their proof of Wagner's Conjecture that graphs are "well-quasi-ordered" under the minor-order, which is the following theorem.

The Graph Minors Theorem ([31]). *Any infinite family of graphs contains two members such that one is isomorphic to a minor of the other.*

This implies that for any minor-closed graph property there are only finitely many *excluded minors*, these are graphs that do not have the property but whose proper minors do have the property. For planarity, for instance, there are exactly two excluded minors: K_5 and $K_{3,3}$; this is Kuratowski's famous characterization of planarity [22].

So, by the Graph Minors Theorem, to test a minor-closed graph property we only need to test containment of each of its excluded minors individually. That this is possible is another crucial outcome of Robertson and Seymour's work.

The Graph Minor Recognition Theorem ([29]). *For each graph H, there exists a polynomial-time algorithm for testing if a graph has a minor isomorphic to H.*

This answered one of the twelve open problems in Garey and Johnson's 1979 book on computational complexity [8].

So minor-closed graph properties can be tested efficiently. However, as noted earlier, also matrices are widely used as modeling tools, for example in integer programming models for operations research. Integer programming models are very general and powerful, but in a sense too general; they lead to "NP-hard" problems. However if the matrix in an integer programming model is *totally unimodular*, that means if all subdeterminants are $0, 1$ or -1, then linear programming methods do solve the problem [20], and these methods are efficient. So this raises the issue of testing totally unimodularity, another open problem back in '79, in Garey and Johnson's book. Now, it turns out that a matrix being totally unimodular means that it is in a certain sense representable over any field, and also this embeddability property is closed under certain minor-operations. So also here the fundamental issue of testing minor-closed properties arises. For that we work at extending Robertson and Seymour's graph minor theory to matrices. As the issues involved do not so much concern the actual matrices, but rather the underlying "matroids", we work in that setting.

A *matroid* consists of a finite set E, the *ground set* of the matroid, and a function r, the *rank function* of the matroid. This rank function is defined on the subsets of E and satisfies the following properties: $0 \le r(X) \le |X|$ for $X \subseteq E$; $r(X) \le r(Y)$ for all $X \subseteq Y \subseteq E$ and $r(X \cup Y) + r(X \cap Y) \le r(X) + r(Y)$ for all $X, Y \subseteq E$. We call $r(X)$ the *rank* of X and $r(E)$ the *rank* of the matroid. The rank function of a matroid M is denoted by r_M. Two matroids are *isomorphic* if there is a rank-preserving bijection between their ground sets.

Matrices yield matroids: If $A = (a_e : e \in E)$ is a matrix with columns a_e over a field $\mathbb{F}$, then the linear rank of the column submatrices $(a_e : e \in F)$ with $F \subseteq E$ is the rank function of a matroid, the *vector matroid* $M(A)$ of A. If a matroid is isomorphic to a vector matroid of a matrix over $\mathbb{F}$, we say that the matroid is *representable over* $\mathbb{F}$ or $\mathbb{F}$-*representable*. (A vector matroid is often described as a configuration of points in a linear, affine or projective space instead of as the collection of columns of a matrix.)

Also graphs yield matroids: Let G be a graph with edge set E and vertex set V. The *rank* of a graph is the number of its vertices minus the number of its components. If $F \subseteq E$, then the *rank* of F is the rank of the subgraph of G with edge set equal to F. This rank yields the rank function of a matroid, the *cycle matroid* $M(G)$ of G. A matroid isomorphic to such a cycle matroid of a graph is called *graphic*. A graphic matroid is representable over any field by a matrix with two non-zero entries in every column, one equal to 1 and one equal to -1. We assume the reader to be familiar with the standard notions from graph theory. For matroid theory we refer to Oxley [25] or Welsh [43], but we will define the matroid terminology we use, as we go.

Now we define matroid minors. Let e be an element of the ground set E of a matroid M. *Deleting* e *from* M is replacing M by the matroid with ground set $E - \{e\}$ and with rank function equal to the restriction of r_M to subsets of $E - \{e\}$. *Contracting* e *from* M is replacing M by the matroid with ground set $E - \{e\}$ and with rank function $r_M(X \cup \{e\}) - r_M(\{e\})$ for each $X \subseteq E - \{e\}$. A *minor* of a matroid is the result of any sequence of deletions and contractions.

A minor of a vector matroid over a field $\mathbb{F}$ is representable over the field as well. Indeed, deleting an element amounts to just deleting the corresponding column whereas contracting an element f amounts to removing a_f from $(a_e : e \in E)$ and projecting all other columns in the direction of a_f on some arbitrary hyperplane not containing a_f. Deletions and contractions in a graph are in one-one correspondence with deletions and contractions in its cycle matroid. Thus the cycle matroid of a minor of a graph is a minor of the cycle matroid of the graph.

So the notion of graph minors is in essence algebraic, or geometric, and in that sense it generalizes to matrices and matroids. This raises the question to what extent Robertson and Seymour's graph minor theory extends to matroids. The following conjecture was by Robertson and Seymour, although to our knowledge not in print.

The Well-Quasi-Ordering Conjecture. Let $\mathbb{F}$ be a finite field. Then any infinite set of $\mathbb{F}$-representable matroids contains two matroids, one of which is isomorphic to a minor of the other.

As yet, the Well-Quasi-Ordering Conjecture has not been resolved for any finite field. Note that it is equivalent to the conjecture that, for a finite field $\mathbb{F}$, any minor-closed class of $\mathbb{F}$-representable matroids has a finite number of $\mathbb{F}$-representable excluded minors.

The finiteness of the field in the Well-Quasi-Ordering Conjecture is essential. Indeed, suppose $\mathbb{F}$ is an infinite field and consider for each integer $n \geq 3$, a $2n \times 3$ matrix with columns $p_1, \ldots, p_n$ and $q_1, \ldots, q_n$, where $p_1, \ldots, p_n$ are vectors in general position in $\mathbb{F}^3$ and each q_i is spanned by p_i and p_{i+1}, but is not spanned by any other pair among $p_1, \ldots, p_n$ (where $p_{n+1} = p_1$). As $\mathbb{F}$ is infinite, such matrices clearly exist. Among the vector matroids of these matrices none is a minor of another. Indeed, all members of the collection have rank 3, so all minors that use a contraction have too low rank to be in the collection; deleting an element from a member of the collection destroys the unique cyclic arrangement of linearly dependent triples p_i, q_i, p_{i+1} in a way that cannot be repaired by further deletions.

We conjecture additionally that for matroids that are representable over a finite field minor-closed properties can be recognized in polynomial time, in other words we conjecture that also the Graph Minor Recognition Theorem extends.

The Minor-Recognition Conjecture. For any finite field $\mathbb{F}$ and any $\mathbb{F}$-representable matroid N, there is a polynomial-time algorithm for testing whether an $\mathbb{F}$-representable matroid contains a minor isomorphic to N.

At the heart of the Graph Minors Project is Robertson and Seymour's Graph Minors Structure Theorem [30]. It describes constructively the graphs that do not contain a given graph as a minor. This constructive description enables techniques to establish the well-quasi-ordering and algorithmic consequences. For matroids, Seymour [36] used this approach successfully for characterizing total unimodularity (see Section 3). Our hope is to use the same strategy for general matroids that are representable over finite fields. Therefore we are developing a structure theory for such matroids. As a major role in the theory of graph minors is played by connectivity, we need an extension of graph connectivity to matroids.

The basic ingredients of graph connectivity are separations, which tell where the connectivity is not that high, and Menger's Theorem, which provides a way of certifying that the connectivity is not that low. A *separation* of a graph G is a pair (G_1, G_2) of subgraphs of G such that $G = G_1 \cup G_2$; the *order* of the separation (G_1, G_2) is the number of vertices of G that lie in both G_1 and G_2. A graph is *k-connected* if it has no separation (G_1, G_2) of order l less than k such that G_1 and G_2 have at least l edges each. One of the fundamental theorems of graph theory is Menger's Theorem.

Menger's Theorem ([24]). *If G is a graph and S and T are two sets of vertices, then there either exist k disjoint paths, each connecting a vertex in S with a vertex in T, or (exclusively) G has a separation (G_1, G_2) of order less than k such that S lies in G_1 and T in G_2.*

As an illustration of how this theorem plays a role in finding minors consider the following easy result of Dirac [4]: *A* 3*-connected graph with at least* 4 *vertices has a minor isomorphic to K_4*. (K_n denotes the complete graph with n vertices; *complete* means that every pair of vertices is connected by an edge.) Here is a proof of Dirac's result: we may assume the graph has two non-adjacent vertices s and t, otherwise the graph is complete and we are done. Apply Menger's Theorem to the set S of neighbours of s and the set T of neighbours of t. This yields three P_1, P_2 and P_3 from s to t that only meet at their ends. As G is 3-connected, $G - s - t$ is connected, so there exists a path Q that connects two of these paths and misses the third path. The union of P_1, P_2, P_3 and Q clearly has a minor isomorphic to K_4. So Dirac's result follows. This is a very easy result of course, but it may convince the reader of the need of a notion of matroid connectivity and a matroidal version of Menger's Theorem.

A *separation* of a matroid M is a partition (X, Y) of the ground set E. The *order* of the separation (X, Y) is $r_M(X) + r_M(Y) - r_M(E) + 1$; for a representable matroid this is the dimension of the intersection of the subspaces spanned by X and Y plus 1. A matroid is *k-connected* if it has no separation (X, Y) of order l less than k such that X and Y have size l or more. If (G_1, G_2) is a separation of G where E_1 and E_2 are the edge sets of G_1 respectively G_2, then (E_1, E_2) is a separation of $M(G)$. When G_1, G_2, and G are connected graphs, the orders of these separations are the same. This is enough to consider matroid separations and matroid connectivity as genuine generalizations of these notions for graphs. And there is a matroidal generalization of Menger's Theorem as well.

Tutte's Linking Theorem ([42]). *Let M be a matroid and X and Y be disjoint subsets of its ground set. Then there exists a minor of M in which (X, Y) is a separation of order at least k or (exclusively) M has a separation (A, B) of order less than k with X in A and Y in B.*

This follows quite easily from Edmonds' Matroid Intersection Theorem [7], which is one of the fundamental theorems of matroid theory. So we see that the basic theory of graph connectivity does extend quite well to matroids.

Besides the fact that a matroid structure theory will help proving the Well-Quasi-Ordering Conjecture and Minor-Recognition Conjecture for matroids, we also expect that it will provide a handle on the following conjecture, probably the most famous open question in matroid theory.

Rota's Conjecture ([33]). Let $\mathbb{F}$ be a finite field. There are, up to isomorphism, only finitely many excluded minors for the class of $\mathbb{F}$-representable matroids.

This also has computational relevance, namely for the question how hard it is to decide if a matroid given by an oracle for the rank function is representable over a field $\mathbb{F}$. Unfortunately, it does take exponentially many oracle calls to decide this, for any field [38]. But if Rota's conjecture is true, then for every finite field $\mathbb{F}$ there exists, for every non-$\mathbb{F}$-representable matroid, a polynomial-length certificate for this non-representability, that requires only a constant number of oracle calls. (It is known that this can be done by a quadratic number of oracle calls [14].)

For the three smallest fields all excluded minors for representability over the field are known: there is one for GF(2) (Tutte [40]), there are four for GF(3) (Bixby [1] and Seymour [37], independently), and there are seven for GF(4) (Geelen, Gerards and Kapoor [9]). For all other finite fields, Rota's conjecture is still open. A structure theory could well provide a way to prove it.

Also in Rota's conjecture finiteness of the field is essential. Lazarson [23] showed that there are an infinite number of excluded minors for representability over the reals and this is certainly true for all other infinite fields.

Summarizing, en route for these three conjectures we are working at establishing the structure of minor-closed proper subclasses of matroids representable over a finite field. This work has already had some success. It turns out that excluding the cycle matroid of a planar graph as a minor imposes tangible structure on matroids over finite fields, so we begin with discussing that.

2. Excluding a planar graph

Let $\mathbb{F}$ be a finite field and let H be a planar graph. We give a constructive structural description of $\mathbb{F}$-representable matroids with no $M(H)$-minor and show that this description enables significant progress on the three conjectures in Section 1. Essentially the structure is to be decomposable into small pieces along low-order separations. We will first explain what that means.

A *branch-decomposition* of matroid M is a tree T in which all vertices have degree 1 or 3, where the degree-1 vertices of T are in 1-1 correspondence with the elements of the ground set E of M. The *width* of an edge e in T is the order of the separation (X, Y) of M where X contains the elements of E that correspond to the degree-1 vertices of T in one component of $T - e$ and Y the elements of E that correspond to the degree-1 vertices of T in the other component of $T - e$. So a branch-decomposition is a data-structure for a collection of separations. The *width* of a branch-decomposition is the maximum of the widths of its edges and the *branch-width* of a matroid is the minimum of the widths of all its branch-decompositions. So roughly low branch-width means to be decomposable into small pieces along low-order separations.

Branch-width is a matroid generalization of branch-width for graphs defined by Robertson and Seymour [28]. For graphs it is, up to a constant multiplicative bound, the same as tree-width, also introduced by Robertson and Seymour. In the Graph Minors Project they mainly use tree-width and that notion does extend to matroids [19] as well. But branch-width is easier to work with for matroids, so here we will only use branch-width, also for graphs. Robertson and Seymour prove the following result.

The Grid Theorem for graphs ([27]). *For each planar graph H there is an integer k such that any graph with branch-width at least k has a minor isomorphic to H.*

This result is called the Grid Theorem because, as every planar graph is a minor of a grid, it suffices to prove it for the case that H is a grid. Here, *a grid*, or rather *an n by n grid*, refers to the graph with a vertex (i, j) for each pair of integers i and j between 1 and n and an edge between any two pairs (i_1, j_1) and (i_2, j_2) with $|i_1 - i_2| + |j_1 - j_2| = 1$. If a matroid has a minor isomorphic to a cycle matroid of an n by n grid, we say it has a *grid-minor*. To convince oneself that each planar graph H is a minor of a sufficiently large grid, visualize H as drawn without crossings and with the edges and vertices as thick lines and dots on a piece of grid paper with a very fine grid.

Consider a class of graphs that do not have a fixed planar graph as a minor. By the Grid Theorem for graphs, the members of that class have bounded branch-width. This constructive characterization provides considerable traction for both algorithmic and structural problems. For example, Robertson and Seymour [26] prove that any class of graphs of bounded branch-width is well-quasi-ordered. This extends to matroids over finite fields.

Theorem ([12]). *Let $\mathbb{F}$ be a finite field and k an integer. Then each infinite set of $\mathbb{F}$-representable matroids with branch-width at most k has two members such that one is isomorphic to a minor of the other.*

Johnson, Robertson, and Seymour [21] conjectured that also the Grid Theorem for graphs extends to matroids over finite fields and this is indeed the case.

The Grid Theorem for matroids ([13]). *For each finite field $\mathbb{F}$ and each planar graph H, there exists an integer k such that each $\mathbb{F}$-representable matroid with branch-width at least k has a minor isomorphic to $M(H)$.*

As a consequence we obtain the following partial result towards the Well-Quasi-Ordering Conjecture.

Corollary. *Let $\mathbb{F}$ be a finite field and H a planar graph. Then any infinite set of $\mathbb{F}$-representable matroids with no minor isomorphic to $M(H)$ contains two matroids such that one is isomorphic to a minor of the other.*

In combination with results of Hliněný [18], we also obtain partial progress towards the Minor-Recognition Conjecture.

Corollary. *For each finite field $\mathbb{F}$ and each planar graph H, there is a polynomial-time algorithm for testing whether or not an $\mathbb{F}$-representable matroid contains a minor isomorphic to $M(H)$.*

So for matroids over a fixed finite field we can efficiently test all minor-closed properties that do not hold for the cycle matroid of all planar graphs.

Geelen and Whittle [17] show that for a finite field $\mathbb{F}$ and integer k, the number of excluded minors for $\mathbb{F}$-representability that have branch width at most k is finite. In combination with the Grid Theorem for matroids this yields the following result.

Corollary. *For each finite field $\mathbb{F}$ and each planar graph H, there are only finitely many excluded minors for $\mathbb{F}$-representability that do not have $M(H)$ as a minor.*

We see that the structure imposed on a class of matroids by excluding the matroid of a planar graph as a minor yields restricted solutions to the Well-Quasi-Ordering Conjecture, the Minor-Recognition Conjecture, and Rota's Conjecture, and that is a promising beginning.

The Grid Theorem for matroids is absolutely central in developing a structure theory for matroids. When specialized to graphs, the proof in [13] is different from the existing proofs in [3], [27], [32]. It is important to note that we had access to an extraordinary 150-page handwritten manuscript [21] of Johnson, Robertson, and Seymour describing their progress towards a grid theorem for matroids. The techniques we learned from their manuscript played a crucial role in parts of our proof. The proof also makes use of earlier results we obtained together with Neil Robertson [10], [11].

Regarding the result above on well-quasi-ordering of $\mathbb{F}$-representable matroids of bounded branch-width it is interesting to note that the finiteness of $\mathbb{F}$ is essential there. This is illustrated by the sequence of matrices given in Section 1, below the Well-Quasi-Ordering Conjecture; they all have branch-width at most 3, as they all have rank 3. On the other hand, there are only finitely many excluded minors for the class of all matroids of branch width at most k, representable or not [11].

We conclude this section with a comment regarding Rota's Conjecture. We have seen that for every finite field GF(q) there are a finite number of excluded minors for GF(q)-representability of any given branch width. In [15] it is proved that an excluded minor for GF(q)-representability of sufficiently large branch width cannot contain a PG($q+6, q$)-minor. (PG(n, q) is the matroid represented by points of the projective geometry of order n over GF(q).) So it follows that if Rota's Conjecture fails for GF(q), then there must exist excluded minors with arbitrarily large grid-minors and no large projective geometry as a minor.

3. An example: the structure of regular matroids

With the results in the previous section in hand we proceed towards a structure theory for matroids over finite fields. One of the prototypes of structural matroid theory and its algorithmic consequences concerns the totally unimodular matrices mentioned in Section 1. The question if a certain given matrix is totally unimodular can be translated into the question if a related, easy-to-construct, GF(2)-representable matroid is representable over all fields. Such matroids are called *regular*. Regularity is a minor-closed property. Tutte [40] proved that a GF(2)-representable matroid is regular if and only if it does not have a minor isomorphic to PG(2, 2), also called the *Fano matroid*, or to the dual of PG(2, 2). Here the *dual* of a matroid M is the matroid M^* with the same ground set E as M and with rank-function $r_{M^*}(X) = |X| - r_M(E-X) + r_M(E)$. Representability over a field $\mathbb{F}$ is closed under duality, hence so is regularity. Taking minors commutes with duality; although the roles of deletion and contraction swap.

Tutte's excluded minor characterization of regular matroids is one of the gems of matroid theory, but it does not tell how to decide if a given GF(2)-representable matroid is regular or not. That question was answered by a structural result, Seymour's Regular Matroid Decomposition Theorem [36]: *A matroid over* GF(2) *is regular if and only if it is the* 1-, 2- *or* 3-*sum of graphic matroids, duals of graphic matroids and copies of a particular* 10-*element matroid called* R_{10}. Here a 1-, 2- or 3-sum of two (representable) matroids is carried out by embedding each of them in a distinct projective space and then combining these projective spaces by taking either their direct sum, in case of a 1-sum, or identifying single points or lines, in case of a 2-sum or a 3-sum. These "meeting" points or lines should be in both matroids and may or may not be deleted from the matroid after the composition. The sums as well as the reversed "decomposition" operations preserve regularity.

So Seymour's result gives a structural description of the class of regular matroids. Their "global structure" is that they are composed from smaller pieces along low-order separations. The pieces sit together in a tree-like fashion. The description of these pieces provides the "local structure": each piece is either a graphic matroid, the dual of a graphic matroid or isomorphic to R_{10}. This combination of global and local structure is typical for all structural results in this paper.

Seymour's structural characterization of regular matroids is constructive, it can be used to design an algorithm for testing regularity in polynomial time. This goes as follows. First decide if the matroid is a 1-, 2- or 3-sum of smaller matroids. This can be done in polynomial time, as gluing two matrices together leaves a separation of order at most 3 in the composed matroid and Cunningham and Edmonds [2] observed that detecting these is a matroid intersection problem, which is solvable in polynomial time (Edmonds [7]). When the matroid is fully decomposed into "4-connected pieces", each piece is tested for being isomorphic to R_{10}, which is trivial, or being a graphic matroid or the dual of a graphic matroid, which can be done by Tutte's polynomial-time algorithm for testing graphicness [41]. If all pieces pass the test, the original matroid is regular, otherwise it is not.

By the relation between regularity and total unimodularity this yields an algorithm for testing if a real matrix is totally unimodular or not (see Schrijver [35, Chapter 20] for a description of this algorithm in terms of the matrices). This is the only known polynomial-time algorithm for testing totally unimodularity. Thus the structure of matroids is crucial for the algorithmic aspects of this central property in operations research and combinatorial optimization. Actually matroids in general do play a major role in the theory of combinatorial optimization, see Schrijver [34]. A book on matroid decomposition is Truemper [39], it mainly concerns regular matroid decomposition and related topics.

With Seymour's regular matroid decomposition in mind we next discuss what we expect to be the structure of minor-closed classes of matroids that are representable over a finite field. It should be noted that the results will not be as "tight" as in Seymour's decomposition theorem. Seymour provided a constructive description of all binary matroids that contain neither the Fano matroid nor its dual as a minor. More-over, none of the matroids obtained via that construction contain the Fano matroid or its dual. In contrast, the Grid Theorem for graphs provides a construction for the graphs that do not contain a given planar graph H as a minor, but some graphs obtained via the construction may contain H as a minor. The construction is, however, sufficiently restrictive that it does not build all planar graphs. For algorithmic and well-quasi-ordering purposes, this is good enough.

4. Global structure and local structure

In Section 2 we discussed the structure of classes of matroids over finite fields that do not have a minor isomorphic to the cycle matroid of a particular planar graph: they can be decomposed into small pieces along low-order separations; they have low branch-width. So explorations beyond that concern matroids with high branch-width. The existence of large grid-minors in such matroids is useful in investigating their structure, but a matroid may have several high branch-width parts that are separated by low-order separations and we have to describe the structure of these parts separately. To get a handle on these parts, Robertson and Seymour [28] introduce tangles. A tangle really just indicates for each low-order separation on which side a particular high branch-width part lies. Formally, a *tangle of order t* assigns to each separation (X, Y) of order less than t one of X and Y as the *small side of* (X, Y) and the other side as the *big side of* (X, Y). It is required that no three small sides of the tangle cover the ground set of the matroid and that no singletons are big. It turns out that the maximum order of a tangle in a matroid is the branch-width of the matroid; for graphs this was shown by Robertson and Seymour [28] and for matroids by Dhamatilike [5] (although this result was implicit in [28]).

Combining this with the Grid Theorem, we see that grid-minors yield tangles. Indeed, if $F \subseteq M$ is the set of elements of an n by n grid-minor of M, then F partitions naturally in the "horizontal" and "vertical" *lines* of the grid. If we consider for each separation (X, Y) of order less than n the side that contains a line in F as big, then that yields a tangle of order n. It was shown in [16] that for any finite field $\mathbb{F}$ and any n there exists an integer t such that in any $\mathbb{F}$-representable matroid any tangle of order t controls an n by n grid-minor. This generalizes a result from [32] for graphs. Here a tangle *controls* a minor N of M if no small side of a separation of order less than the rank of N contains all elements of N.

So tangles "locate" highly connected areas of the matroid. If all small sides of one tangle are small in some other tangle, then they both seem to refer to the same highly connected part but the latter tangle does that more accurately. Therefore we are mainly interested in the *maximal* tangles, those for which the collection of small sets is inclusion-wise maximal. It turns out that matroids, like graphs [28], can be viewed as consisting of their maximal tangles put together in a tree-shaped structure, see [16] for details. This provides a global picture of a matroid. To complete that picture we have to describe the individual tangles, the local structure.

To explain what we mean with that, we first explain what it means to reduce a set S in an $\mathbb{F}$-represented matroid M. Consider M as a collection E of points in a projective geometry. Let X be the span of $E - S$ in that projective geometry. For each $S' \subseteq S$ whose span meets X in a single point, we call that point $x_{S'}$. Let Y be the set of all points $x_{S'}$ for all such sets S'. Replacing M by the matroid represented by the union of Y and $E - S$ is called *reducing* S. (For graphs, this more or less means to remove the edges in S and to add an edge between any pair of vertices that both lie in S and in the complement of S.)

Let $\mathcal{C}$ be a class of matroids. A tangle has *local structure in* $\mathcal{C}$ if there exist separations $(S_1, B_1), \dots, (S_k, B_k)$ in M with disjoint small sides $S_1, \dots, S_k$ such that the matroid obtained from M by reducing each of $S_1, \dots, S_k$ is in $\mathcal{C}$. To describe the full structure we only need to characterize the minor-closed classes $\mathcal{C}$ that provide the local structure of tangles in matroids over the finite field that do not contain a particular minor.

5. The local structure of graph tangles

The Graph Minor Structure Theorem says that for any n there exists a surface Σ and integers m, d, k such that the tangles of a graph with no minor isomorphic to K_n have local structure in the class of graphs that lie on a surface Σ with m vortices of depth at most d and k extra vertices. We explain what this means.

A *vortex with connectors* $v_1, \dots, v_p$ is a graph H that is the union of graphs $H_1, \dots, H_p$ such that v_i is a vertex in H_i for each $i = 1, \dots, p$ and such that if a vertex v of H occurs in H_i and H_j for some $i, j = 1, \dots, p$ then v either occurs in all of $H_{i+1}, \dots, H_{j-1}$ or in all of $H_{j+1}, \dots, H_{i-1}$ (indices modulo n). The maximum size of the subgraphs $H_1, H_2, \dots, H_p$ is the *depth* of the vortex.

A graph is *on a surface* Σ *with m vortices of depth at most d* if it can be constructed as follows: take a graph drawn on Σ, select m faces and add to each of these faces a vortex of depth d to G that meets G and the other added vortices only in its connectors $v_1, \dots, v_n$ which lie in that order around the boundary of the face. If we additionally add k new vertices and new edges from these vertices to each other and to the rest of the graph, we obtain *a graph that lies on a surface* Σ *with m vortices of depth at most d and k extra vertices*.

6. The local structure of matroid tangles

What are the minor-closed classes needed to describe the local structure of matroids that are representable over a finite field? One natural minor-closed class is the class of graphic matroids. Also, if $\mathbb{F}'$ is a subfield of $\mathbb{F}$, then the class of $\mathbb{F}'$-representable matroids is a minor-closed class of $\mathbb{F}$-representable matroids. There is another natural class, of Dowling matroids. They are like graphs and originally introduced by Dowling [6] and studied in greater depth by Zaslavsky [44], [45].

A *Dowling matroid* is a matroid that can be represented over a field $\mathbb{F}$ by a matrix with the property that every column has at most two non-zero elements. We call such matrix a *Dowling representation* of the matroid. If the ratio between the non-zero elements in each column of a Dowling representation is in a subgroup Γ of the multiplicative group of $\mathbb{F}$, we call the matroid *a Dowling matroid over* Γ. One can naturally associate a graph $G(A)$ with a Dowling representation A. Each row of A is a vertex of $G(A)$ and each column of A with two non-zeroes yields an edge in $G(A)$ connecting the vertices corresponding to the rows that have the non-zeroes in that column. Thus we get for each surface and each subgroup of the multiplicative group of $\mathbb{F}$ the class of $\mathbb{F}$-representable Dowling matroids that have Dowling representations over the subgroup and whose associated graphs embed on the surface. Obviously such a class is minor-closed.

In fact, we can extend such minor-closed class by allowing a bounded number of "vortices" of bounded depth, these are obtained by adding matroid elements into bounded-rank subspaces arranged in a cyclic manner around a face in the embedding, similar to vortices in graphs.

Finally we can extend a minor-closed class $\mathcal{C}$ of matroids by considering for some integer k the class of all rank-l perturbations of the members of $\mathcal{C}$ with $l \leq k$. Here an $\mathbb{F}$-representable matroid M is a *rank-l perturbation* of an $\mathbb{F}$-representable matroid N if M and N have representations A and B, respectively, with the linear rank of $A - B$ equal to l.

Splitting a vertex in a graph amounts to a rank-1 perturbation of its cycle matroid. So adding k vertices to a graph amounts to adding a single vertex followed by a rank-$(k-1)$ perturbation of the cycle matroid. Adding a single vertex to a graph G does in general not correspond to a low-rank perturbation. However, fortunately, the cycle matroid of the resulting graph has a Dowling representation A with $G(A) = G$. Hence the Graph Minors Structure Theorem is captured by the matroid classes given above.

Now we state our main results and conjectures on the structure of minor-closed classes over a finite field GF(q), where $q = p^k$ for some fixed prime p and some fixed integer k. We distinguish between three types of minor-classes. The first type are the classes that do not contain the cycle matroid of large complete graphs nor their duals. The second type are the classes that do not contain large projective geometries over the prime field GF(p) of GF(q). The third type are the classes that do not contain large projective geometries over GF(q). In each of the cases, $\mathcal{T}$ is a tangle in a GF(q)-representable matroid.

Below n is a fixed integer and each of the qualitative bounds "low", "bounded", or "sufficiently large" indicates a bound only depending on q and n, so not on the particular tangles or matroids.

Excluding $M(K_n)$ and $M(K_n)^*$. We believe that we have proved that if $\mathcal{T}$ has sufficiently large order and does not control a minor isomorphic to $M(K_n)$ or to $M(K_n)^*$, then $\mathcal{T}$ has local structure in the class of low-rank perturbations of GF(q)-

representable matroids that can be obtained by adding a bounded number of vortices of bounded depth to a Dowling matroid whose associated graph is embedded in a surface of low genus, or of the duals of such matroids. This implies the Graph Minors Structure Theorem.

With this result and duality, we can now restrict our attention to tangles that control the cycle matroid of a large complete graph.

Excluding PG(n, p). We conjecture that if $\mathcal{T}$ controls a minor isomorphic to $M(K_m)$ for a sufficiently large integer m but $\mathcal{T}$ does not control a minor isomorphic to $\mathrm{PG}(n, p)$, then $\mathcal{T}$ has local structure in the class of low-rank perturbations of $\mathrm{GF}(q)$-representable Dowling matroids.

Roughly speaking the conjectures above state that if M is a $\mathrm{GF}(q)$-representable matroid with no minor isomorphic to $\mathrm{PG}(n, p)$, then M admits a tree-like decomposition such that each part is either essentially a Dowling matroid or is essentially the dual of a Dowling matroid. For a field of prime order this would give the required constructive structural characterization of the minor-closed proper subclasses of matroids representable over the field.

It is interesting to note here that a slight extension of Seymour's regular matroid decomposition says that if a GF(2)-representable matroid has no minor isomorphic to PG(2, 2) then it can be constructed from graphic matroids, their duals, and copies of R_{10} and copies of the dual of PG(2, 2), by 1-, 2- and 3-sums [36]. As graphic matroids are GF(2)-representable Dowling matroids and as R_{10} and the dual of PG(2, 2) are low-rank perturbations of a trivial matroid, this result of Seymour's implies the conjecture above for case that $q = 2$ and $n = 2$.

Excluding PG(n, q). For the case that q is not prime, we conjecture that if $\mathcal{T}$ controls a minor isomorphic to $\mathrm{PG}(m, p)$ for a sufficiently large integer m but $\mathcal{T}$ does not control a minor isomorphic to $\mathrm{PG}(n, q)$, then $\mathcal{T}$ has local structure in the class of $\mathrm{GF}(q)$-representable low-rank perturbations of matroids that are representable over a proper subfield of $\mathrm{GF}(q)$.

Finally we can summarize all of the above into a single conjecture. For any minor-closed proper subclass $\mathcal{M}$ of $\mathrm{GF}(q)$-representable matroids, each matroid in $\mathcal{M}$ admits a tree-like decomposition such that each part is either essentially a Dowling matroid, or is essentially the dual of a Dowling matroid, or is essentially represented over a proper subfield of $\mathrm{GF}(q)$.

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Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Ontario, Canada
E-mail: jfgeelen@uwaterloo.ca

Centrum voor Wiskunde en Informatica, Amsterdam, The Netherlands
and
Technische Universiteit Eindhoven, Eindhoven, The Netherlands
E-mail: bert.gerards@cwi.nl

School of Mathematical and Computing Sciences, Victoria University, Wellington, New Zealand
E-mail: geoff.whittle@vuw.ac.nz