Finite element algorithms for transport-diffusion problems: stability, adaptivity, tractability

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Abstract. Partial differential equations with nonnegative characteristic form arise in numerous mathematical models of physical phenomena: stochastic analysis, in particular, is a fertile source of equations of this kind. We survey recent developments concerning the finite element approximation of these equations, focusing on three relevant aspects: (a) stability and stabilisation; (b) *hp*-adaptive algorithms driven by residual-based *a posteriori* error bounds, capable of automatic variation of the granularity h of the finite element partition and of the local polynomial degree p; (c) complexity-reduction for high-dimensional transport-diffusion problems by stabilised sparse finite element methods.

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1. Introduction

Let Ω be a bounded and simply-connected open set in $\mathbb{R}^d$, $d \geq 2$, with Lipschitz continuous boundary $\partial\Omega$. On Ω, we consider the partial differential equation

$$\mathcal{L}u := -\nabla \cdot (a\nabla u) + \nabla \cdot (bu) + cu = f, \tag{1}$$

where $f \in \mathrm{L}_2(\Omega)$ and $c \in \mathrm{L}_\infty(\Omega)$ are real-valued functions, $b = \{b_i\}_{i=1}^d$ is a vector function whose entries b_i are Lipschitz continuous real-valued functions on $\overline{\Omega}$. We shall, further, assume that $a = \{a_{ij}\}_{i,j=1}^d$ is a *symmetric* matrix whose entries a_{ij} are bounded, Lipschitz continuous real-valued functions defined on $\overline{\Omega}$, and that the matrix a is positive semidefinite, almost everywhere on $\overline{\Omega}$, i.e.,

$$\alpha(\xi) := \xi^\top a(x)\xi \geq 0 \quad \text{for all } \xi \in \mathbb{R}^d \text{ and a.e. } x \in \overline{\Omega}. \tag{2}$$

Under hypothesis (2), the equation (1) is referred to as a *partial differential equation with nonnegative characteristic form*. Equations of this kind frequently arise as mathematical models in physics and chemistry [40] (e.g. in the kinetic theory of polymers [7], [44], [49] and coagulation-fragmentation problems [43]). They also appear in molecular biology [21], population genetics (e.g. in mathematical models of random

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genetic drift) and in mathematical finance. Important special cases of these equations include the following: (a) when the diffusion matrix $a = a^\top$ is positive definite, (1) is an elliptic partial differential equation; (b) when $a \equiv 0$ and the transport direction $b \neq 0$, the partial differential equation (1) is a first-order hyperbolic equation; (c) when $b = (0, \ldots, 0, 1)^\top \in \mathbb{R}^d$ and

$$a = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}$$

where α is a $(d-1) \times (d-1)$ symmetric positive definite matrix, (1) is a parabolic partial differential equation, with time-like direction b. The family of partial differential equations with nonnegative characteristic form also includes a range of other linear second-order partial differential equations, such as degenerate elliptic and ultra-parabolic equations. Furthermore, by a result of Hörmander [35] (cf. Theorem 11.1.10 on p. 67), second-order hypoelliptic operators with constant coefficients have nonnegative characteristic form, after possible multiplication by -1.

For classical types of partial differential equations, such as those under (a), (b) and (c) above, rich families of reliable, stable and accurate numerical techniques have been developed. Yet, there have only been isolated attempts to date to explore computational aspects of the class of partial differential equations with nonnegative characteristic form as a whole (cf. [30] and [33]). In particular, only a limited amount of research has been devoted to the construction and mathematical analysis of adaptive finite element algorithms for these equations; similarly, there has been very little work on the finite element approximation of high-dimensional partial differential equations with nonnegative characteristic form (cf. [57]).

The aim of this paper is to present a brief survey of some recent results in these directions. In Section 2, we state the weak formulation of a boundary-value problem for equation (1). In Sections 3 and 4, we give an overview of stabilised continuous and discontinuous finite element approximations to these equations; we shall also address the question of residual-based a posteriori error estimation for hp-version discontinuous Galerkin approximations of these equations, as well as the construction of sparse stabilised finite element methods for *high-dimensional* partial differential equations with nonnegative characteristic form (Section 5).

2. Boundary conditions and weak formulation

For the sake of simplicity of presentation, we shall assume that Ω is a bounded open polytope in $\mathbb{R}^d$ and we denote by Γ the union of its $(d-1)$-dimensional open faces; clearly, $\Gamma \subset \partial\Omega$ with strict inclusion. The equation (2) will be supplemented by boundary conditions. For this purpose, let $\nu(x) = \{\nu_i(x)\}_{i=1}^d$ denote the unit outward normal vector to Γ at $x \in \Gamma$. On introducing the *Fichera function* (cf. [48])

$x \in \Gamma \mapsto \beta(x) := (b \cdot \nu)(x) \in \mathbb{R}$, we define the following subsets of Γ:

$$\Gamma_0 = \{x \in \Gamma : \ \alpha(\nu(x)) > 0\},$$
$$\Gamma_- = \{x \in \Gamma \backslash \Gamma_0 : \ \beta(x) < 0\}, \quad \Gamma_+ = \{x \in \Gamma \backslash \Gamma_0 : \ \beta(x) \geq 0\}.$$

The set Γ_0 is the *elliptic part* of Γ, while $\Gamma_- \cup \Gamma_+$ represents the *hyperbolic part* of Γ. The sets Γ_- and Γ_+ will be referred to as the hyperbolic *inflow* and *outflow* boundary, respectively. Clearly, $\Gamma = \Gamma_0 \cup \Gamma_- \cup \Gamma_+$. If $\Gamma_0 \neq \emptyset$ and has positive $(d-1)$-dimensional Hausdorff measure $\mathcal{H}^{d-1}(\Gamma_0)$, we shall further decompose it into disjoint subsets Γ_D and Γ_N whose union is Γ_0, with $\mathcal{H}^{d-1}(\Gamma_D) > 0$. We supplement the partial differential equation (1) with the boundary conditions

$$u = g_D \quad \text{on } \Gamma_D \cup \Gamma_-, \qquad \nu \cdot (a \nabla u) = g_N \quad \text{on } \Gamma_N, \tag{3}$$

and adopt the (physically reasonable) hypothesis that $\beta(x) \geq 0$ for a.e. $x \in \Gamma_N$, whenever Γ_N is nonempty. In addition, we assume that the following (standard) positivity hypothesis holds: there exists a positive constant $\hat{c}_0$ such that

$$c(x) + \tfrac{1}{2} \nabla \cdot b(x) \geq \hat{c}_0^2 \quad \text{a.e. } x \in \Omega, \tag{4}$$

and define $c_0 = (c + \frac{1}{2} \nabla \cdot b)^{1/2}$ on $\overline{\Omega}$. Now, consider the following boundary-value problem, corresponding to $g_D = 0$ and $g_N = 0$: find u such that

$$\mathcal{L}u \equiv -\nabla \cdot (a \nabla u) + \nabla \cdot (bu) + cu = f \quad \text{in } \Omega, \tag{5}$$
$$u = 0 \quad \text{on } \Gamma_D \cup \Gamma_-, \tag{6}$$
$$\nu \cdot (a \nabla u) = 0 \quad \text{on } \Gamma_N. \tag{7}$$

Function spaces and weak formulation. The classical Sobolev space on Ω of integer order m, $m \geq 0$, will be denoted by $W_q^m(\Omega)$ for $q \in [1, \infty]$; in the case $q = 2$ we write $H^m(\Omega)$ for $W_2^m(\Omega)$; $W_q^0(\Omega)$ is simply $L_q(\Omega)$. $W_p^m(\Omega)$ is equipped with the Sobolev norm $\| \cdot \|_{W_q^m(\Omega)}$ and seminorm $| \cdot |_{W_q^m(\Omega)}$. For the sake of simplicity, we shall write $\| \cdot \|$ instead of $\| \cdot \|_{L_2(\Omega)}$, and $\| \cdot \|_\kappa$ will denote $\| \cdot \|_{L_2(\kappa)}$ for an open subset κ of Ω. We let $\mathcal{V} = \{v \in H^1(\Omega) : \gamma_{0,\partial\Omega}(v)|_{\Gamma_D} = 0\}$ where $\gamma_{0,\partial\Omega}(v)$ signifies the trace of v on $\partial\Omega$, and define the inner product $(\cdot, \cdot)_{\mathcal{H}}$ by

$$(w, v)_{\mathcal{H}} := (a \nabla w, \nabla v) + (w, v) + \langle w, v \rangle_{\Gamma_N \cup \Gamma_- \cup \Gamma_+}.$$

Here $(\cdot, \cdot)$ denotes the L_2 inner product over Ω and $\langle w, v \rangle_S = \int_S |\beta| wv \, ds$, with β denoting the Fichera function $b \cdot \nu$, as before, and $S \subset \Gamma$. We denote by $\mathcal{H}$ the closure of the space $\mathcal{V}$ in the norm $\| \cdot \|_{\mathcal{H}}$ defined by $\|w\|_{\mathcal{H}} := (w, w)_{\mathcal{H}}^{1/2}$. Clearly, $\mathcal{H}$ is a Hilbert space. For $w \in \mathcal{H}$ and $v \in \mathcal{V}$, we now consider the bilinear form $B(\cdot, \cdot) : \mathcal{H} \times \mathcal{V} \to \mathbb{R}$ defined by

$$B(w, v) := (a \nabla w, \nabla v) - (w, \nabla \cdot (bv)) + (cw, v) + \langle w, v \rangle_{\Gamma_N \cup \Gamma_+},$$

and for $v \in \mathcal{V}$ we introduce the linear functional $L\colon \mathcal{V} \to \mathbb{R}$ by $L(v) := (f, v)$. Note, in particular, that by (4),

$$B(v, v) = (a\nabla v, \nabla v) + \|c_0 v\|^2 + \tfrac{1}{2}\langle v, v\rangle_{\Gamma_N \cup \Gamma_- \cup \Gamma_+} \geq K_0 \|v\|^2_{\mathcal{H}} \quad \text{for all } v \in \mathcal{V},$$

where $K_0 = \min\left(\hat{c}_0^2, \frac{1}{2}\right) > 0$. We shall say that $u \in \mathcal{H}$ is a *weak solution* to the boundary-value problem (5), (6) if

$$B(u, v) = L(v) \quad \text{for all } v \in \mathcal{V}. \tag{8}$$

We note that the boundary conditions $u|_{\Gamma_-} = 0$ on the inflow part Γ_- of the hyperbolic boundary $\Gamma \setminus \Gamma_0 = \Gamma_- \cup \Gamma_+$ and the boundary condition $\nu \cdot (a\nabla u) = 0$ on the Neumann part Γ_N of the elliptic boundary Γ_0 are imposed weakly, through (8), while the boundary condition $u|_{\Gamma_D} = 0$ on the Dirichlet part, Γ_D, of Γ_0 is imposed strongly, through the choice of the function space $\mathcal{H}$. The existence of a unique weak solution is guaranteed by the following theorem (cf. also Theorem 1.4.1 on p. 29 of [48] and [57] for a similar result in the special case of $\Gamma_N = \emptyset$; for $\Gamma_N \neq \emptyset$ the proof is identical to that in [57]).

Theorem 2.1. *Suppose that $c_0(x) \geq \hat{c}_0 > 0$ for all $x \in \overline{\Omega}$. Then, for each $f \in L_2(\Omega)$, there is a unique u in a Hilbert subspace $\hat{\mathcal{H}}$ of $\mathcal{H}$ such that* (8) *holds.*

Next, we shall consider the discretisation of the problem (8), first by a stabilised Galerkin method based on continuous piecewise polynomials, and then using discontinuous piecewise polynomials.

3. Continuous piecewise polynomial approximation: the streamline-diffusion method

As in the previous section, we suppose that Ω is a bounded open polytope in $\mathbb{R}^d$, $d \geq 2$. Let $\mathcal{T}_h = \{\kappa\}$ be an admissible subdivision of Ω into open element domains κ which is subordinate to the decomposition of Γ into the subsets Γ_D, Γ_N, Γ_- and Γ_+; here h is a piecewise constant mesh function with $h(x) = h_\kappa = \mathrm{diam}(\kappa)$ when x is in element $\kappa \in \mathcal{T}_h$. We shall assume that each $\kappa \in \mathcal{T}_h$ is the image, under a bijective affine map F_κ, of a fixed *master element* $\hat{\kappa}$, where $\hat{\kappa}$ is either an open unit simplex or an axiparallel open unit hypercube in $\mathbb{R}^d$. We shall also suppose that the family of partitions $\{\mathcal{T}_h\}_{h>0}$ is

(a) *regular* (namely, the closures of any two elements in the subdivision are either disjoint or share a common face of dimension $\leq d-1$); and

(b) *shape-regular* (namely, there exists a positive constant c_1, independent of h, such that $c_1 h_\kappa^d \leq \mathrm{meas}(\kappa)$ for all $\kappa \in \bigcup_h \mathcal{T}_h$).

For $p \geq 1$, we denote by $\mathcal{P}_p(\hat{\kappa})$ the set of polynomials of degree at most p on $\hat{\kappa}$ when $\hat{\kappa}$ is an open unit simplex; when $\hat{\kappa}$ is an axiparallel open unit hypercube, we let $\mathcal{Q}_p(\hat{\kappa})$ denote the set of all tensor-product polynomials of degree at most p in each coordinate direction. We define the *finite element space*

$$\mathcal{H}_{h,p} = \{v \in \mathcal{H} \cap \mathrm{C}(\overline{\Omega}) : v|_\kappa \in \mathcal{R}_p(\kappa) \text{ for all } \kappa \in \mathcal{T}_h\},$$

where $\mathcal{R}_p(\kappa) = \{w \in \mathrm{L}_1(\kappa) \,:\, w \circ F_\kappa \in \mathcal{R}_p(\hat{\kappa})\}$ and $\mathcal{R}_p$ is either $\mathcal{P}_p$ or $\mathcal{Q}_p$.

Next, we formulate the streamline-diffusion finite element approximation of (8). The method was originally introduced by Hughes and Brooks [36] in 1979 for elliptic transport-dominated diffusion equations. Its analysis was pursued by a number of authors (see [38], [39], [54], for example). The definition of the method stems from the empirical observation that standard Galerkin finite element approximations to transport-dominated diffusion problems exhibit nonphysical numerical oscillations which occur predominantly in the direction of subcharacteristic curves (i.e. the characteristic curves of the underlying hyperbolic problem); the standard Galerkin method is therefore supplemented with numerical diffusion/dissipation in the direction of the subcharacteristics through the inclusion of a *streamline-diffusion stabilisation term*. For a survey of recent perspectives on stabilised and multiscale finite element methods for partial differential equations, including transport-dominated diffusion problems, we refer to the survey paper of Brezzi and Marini [14]; see also [13], [15]. Here, we follow the exposition in [33] and consider the bilinear form $B_\delta(\,\cdot\,,\,\cdot\,)$ defined by

$$\begin{aligned} B_\delta(w, v) = (a\nabla w, \nabla v) - (w, \nabla \cdot (bv)) + (cw, v) \\ + \langle w, v\rangle_{\Gamma_\mathrm{N}\cup\Gamma_+} + \sum_{\kappa\in\mathcal{T}_h} (\hat{\mathcal{L}}w, \delta_\kappa\, b\cdot\nabla v)_\kappa \end{aligned}$$

and the linear functional $\ell_\delta(v) = \sum_\kappa (f,\, v+\delta_\kappa\, b\cdot\nabla v)_\kappa$, where, on element $\kappa \in \mathcal{T}_h$, we define $\hat{\mathcal{L}}$ by $\hat{\mathcal{L}}w = -\nabla\cdot(P_\kappa(a\nabla w)) + b\cdot\nabla w + cw$, $w \in \mathrm{H}^1(\kappa)$, and P_κ signifies the orthogonal projection in $[\mathrm{L}_2(\kappa)]^d$ onto $[\mathcal{R}_p(\kappa)]^d$. In these definitions $(\cdot,\cdot)_\kappa$ denotes the L_2 inner product over κ and the nonnegative piecewise constant function δ, called the *streamline-diffusion stabilisation parameter*, is defined by $\delta|_\kappa = \delta_\kappa$ for $\kappa \in \mathcal{T}_h$, where δ_κ is a nonnegative constant on element κ. The precise choice of δ will be discussed below.

Now, the streamline-diffusion finite element method is defined as follows: find $u_\mathrm{SD} \in \mathcal{H}_{h,p}$ such that

$$B_\delta(u_\mathrm{SD}, v) = \ell_\delta(v) \quad \text{for all } v \in \mathcal{H}_{h,p}. \tag{9}$$

Here, we shall focus on the stability and error analysis of this method. A key property is the following: from (8) and (9) we deduce that if $u \in \mathcal{H} \cap \mathrm{H}^2(\Omega)$ then

$$B_\delta((u - u_\mathrm{SD}), v) = \sum_{\kappa\in\mathcal{T}_h} (\nabla\cdot(a\nabla u - P_\kappa(a\nabla u)), \delta_\kappa b\cdot\nabla v)_\kappa \quad \text{for all } v \in \mathcal{H}_{h,p}. \tag{10}$$

In particular, if a is a constant matrix, then the projection operator P_κ can be replaced by the identity operator. In this case the right-hand side in (10) is zero and this identity is then referred to as the *Galerkin orthogonality* property of the streamline-diffusion finite element method (9).

Next we show the stability of the method (9) and state an optimal order *a priori* error bound. The bound will be expressed in terms of the so-called streamline-diffusion norm $|||\cdot|||_{\mathrm{SD}}$ defined by

$$|||v|||_{\mathrm{SD}}^2 = \|\nabla v\|_a^2 + \hat{c}_0^2\|v\|^2 + \|v\|_{\Gamma_{\mathrm{N}}\cup\Gamma_-\cup\Gamma_+}^2 + \|\sqrt{\delta}\, b\cdot\nabla v\|^2,$$

where $\|\nabla v\|_a^2 = (a\nabla v, \nabla v)$. The analysis requires the following results; cf. [33].

Lemma 3.1 (Inverse inequality). *There exists a positive constant $C_{\mathrm{inv}} = C_{\mathrm{inv}}(c_1)$, independent of a, h_κ and p such that*

$$\|\nabla\cdot(P_\kappa(a\nabla v))\|_{\mathrm{L}_2(\kappa)} \le C_{\mathrm{inv}}\,\frac{p^2}{h_\kappa}\,\|a\nabla v\|_{\mathrm{L}_2(\kappa)} \quad \textit{for all } v\in\mathcal{H}_{h,p},\ \kappa\in\mathcal{T}_h.$$

Lemma 3.2. *Suppose that M is a real $d\times d$ symmetric positive semidefinite matrix and let $|\cdot|$ denote the Euclidean norm on $\mathbb{R}^d$; then $|M\xi|^2 \le \rho(M)(M\xi,\xi)$ for all $\xi\in\mathbb{R}^d$, where $\rho(M)=\max_{1\le i\le d}\lambda_i$ is the spectral radius of M and λ_i, $i=1,\dots,d$, are the (real, nonnegative) eigenvalues of M.*

Now we are ready to discuss the coercivity of the bilinear form $B_\delta(\cdot,\cdot)$ over $\mathcal{H}_{h,p}\times\mathcal{H}_{h,p}$. To this end we define $\mathcal{T}_h' = \{\kappa\in\mathcal{T}_h : \|b\|_{\mathrm{L}_\infty(\kappa)}\neq 0\}$.

Proposition 3.3. *Suppose that the streamline-diffusion parameter δ_κ on element κ is selected, with the convention $1/0=\infty$, so that*

$$0\le\delta_\kappa\le\tfrac{1}{2}\min\left(\frac{h_\kappa^2}{(C_{\mathrm{inv}})^2\,\|\rho(a)\|_{\mathrm{L}_\infty(\kappa)}\,p^4},\ \frac{\hat{c}_0^2}{\|c\|_{\mathrm{L}_\infty(\kappa)}^2}\right) \quad \textit{for all } \kappa\in\mathcal{T}_h'. \tag{11}$$

Then, the bilinear form $B_\delta(\cdot,\cdot)$ is coercive on $\mathcal{H}_{h,p}\times\mathcal{H}_{h,p}$, i.e.

$$B_\delta(v,v)\ge\tfrac{1}{2}|||v|||_{\mathrm{SD}}^2 \quad \textit{for all } v\in\mathcal{H}_{h,p}.$$

Proof. Integrating by parts gives

$$\begin{aligned} B_\delta(v,v) \ge{}& (a\nabla v,\nabla v) + \int_\Omega c_0^2 v^2\,\mathrm{d}x + \tfrac{1}{2}\|v\|_{\Gamma_{\mathrm{N}}\cup\Gamma_-\cup\Gamma_+}^2 + \|\sqrt{\delta}\,b\cdot\nabla v\|^2 \\ &- \sum_{\kappa\in\mathcal{T}_h'}\delta_\kappa\left(\|\nabla\cdot(P_\kappa(a\nabla v))\|_\kappa + \|cv\|_\kappa\right)\|b\cdot\nabla v\|_\kappa, \end{aligned} \tag{12}$$

for $v\in\mathcal{H}_{h,p}$. Now, using Lemma 3.1 and Lemma 3.2 with $M=a$ and $\xi=\nabla v$,

$$\|\nabla\cdot(P_\kappa(a\nabla v))\|_\kappa + \|cv\|_\kappa \le C_{\mathrm{inv}}\,\frac{p^2}{h_\kappa}\,\|\rho(a)\|_{\mathrm{L}_\infty(\kappa)}^{1/2}\|\nabla v\|_{a,\kappa} + \|c\|_{\mathrm{L}_\infty(\kappa)}\|v\|_\kappa,$$

with the notation $\|\nabla v\|_{a,\kappa} = (a\nabla v, \nabla v)_\kappa^{1/2}$. Thus, for any real number $\gamma > 0$,

$$\sum_{\kappa\in\mathcal{T}_h'} \delta_\kappa \left(\|\nabla\cdot(P_\kappa(a\nabla v))\|_\kappa + \|cv\|_\kappa\right)\|b\cdot\nabla v\|_\kappa \le \gamma \sum_{\kappa\in\mathcal{T}_h'} \delta_\kappa \|b\cdot\nabla v\|_\kappa^2$$
$$+ \frac{1}{2\gamma}\sum_{\kappa\in\mathcal{T}_h'} \delta_\kappa \|c\|_{\mathrm{L}_\infty(\kappa)}^2 \|v\|_\kappa^2 + \frac{1}{2\gamma}\sum_{\kappa\in\mathcal{T}_h'} \delta_\kappa \,(C_{\mathrm{inv}})^2 \,\|\rho(a)\|_{\mathrm{L}_\infty(\kappa)} \frac{p^4}{h_\kappa^2}\,\|\nabla v\|_{a,\kappa}^2.$$

Choosing $\gamma = 1/2$, we deduce from (12) and the definition of c_0 that

$$B_\delta(v,v) \ge \|\nabla v\|_a^2 + \hat{c}_0^2\|v\|^2 + \tfrac{1}{2}\|v\|_{\Gamma_{\mathrm{N}}\cup\Gamma_-\cup\Gamma_+}^2 + \tfrac{1}{2}\|\sqrt{\delta}\; b\cdot\nabla v\|^2$$
$$- \sum_{\kappa\in\mathcal{T}_h'} \delta_\kappa \|c\|_{\mathrm{L}_\infty(\kappa)}^2\|v\|_\kappa^2 - \sum_{\kappa\in\mathcal{T}_h'} \delta_\kappa \,(C_{\mathrm{inv}})^2 \,\|\rho(a)\|_{\mathrm{L}_\infty(\kappa)} \frac{p^4}{h_\kappa^2}\,\|\nabla v\|_{a,\kappa}^2.$$

Selecting the streamline-diffusion parameter as in (11), the result follows. □

Corollary 3.4 (Stability). *Under the hypotheses of Proposition* 3.3,

$$|||u_{\mathrm{SD}}|||_{\mathrm{SD}}^2 \le 4\hat{c}_0^{-2}\|f\|^2 + 4\sum_{\kappa\in\mathcal{T}_h'} \delta_\kappa \|f\|_\kappa^2.$$

In particular, if $f = 0$ then $u_{\mathrm{SD}} = 0$; since $\mathcal{H}_{h,p}$ is a finite-dimensional linear space, it follows that (9) has a unique solution $u_{\mathrm{SD}} \in \mathcal{H}_{h,p}$ for any $f \in \mathrm{L}_2(\Omega)$. The next result concerns the accuracy of the method (9).

Theorem 3.5. *Let the streamline-diffusion parameter δ_κ be chosen so that*

$$0 < \delta_\kappa \le \frac{1}{2}\min\left(\frac{h_\kappa^2}{(C_{\mathrm{inv}})^2\,\|\rho(a)\|_{\mathrm{L}_\infty(\kappa)}\,p^4}\,,\ \frac{\hat{c}_0^2}{\|c\|_{\mathrm{L}_\infty(\kappa)}^2}\right) \quad \textit{for all } \kappa \in \mathcal{T}_h',$$

with the convention $1/0 = \infty$. Then, assuming that $u \in \mathcal{H}\cap \mathrm{H}^k(\Omega)\cap \mathrm{C}(\overline{\Omega})$, with a positive integer k and $a \in [\mathrm{W}_\infty^{k-1}(\kappa)]^{d\times d}$, $\kappa \in \mathcal{T}_h$, the following error bound holds:

$$|||u - u_{\mathrm{SD}}|||_{\mathrm{SD}} \le C\Big(\sum_{\kappa\in\mathcal{T}_h} \frac{h_\kappa^{2\tau-1}}{p^{2k-1}} M_\kappa(a,b,c,h_\kappa,p)\|u\|_{\mathrm{H}^k(\kappa)}^2\Big)^{1/2},$$

where $\tau = \min(p+1,k)$, C is a positive constant which depends only on c_1 and k, $M_\kappa(a,b,c,h_\kappa,p) = A_\kappa(p/h_\kappa) + B_\kappa + C_\kappa(h_\kappa/p)$, with

$$A_\kappa = \begin{cases} \|a\|_{\mathrm{L}_\infty(\kappa)} + \dfrac{\|a\|_{\mathrm{L}_\infty(\kappa)}^2 + \|a\|_{\mathrm{W}_\infty^{k-1}(\kappa)}^2}{\|\rho(a)\|_{\mathrm{L}_\infty(\kappa)}}, & \textit{when } \|\rho(a)\|_{\mathrm{L}_\infty(\kappa)} \ne 0,\\ 0, & \textit{otherwise},\end{cases}$$

$$B_\kappa = \begin{cases} \|b\|_{\mathrm{L}_\infty(\kappa)}(1 + D_\kappa + D_\kappa^{-1}), & \textit{when } \|b\|_{\mathrm{L}_\infty(\kappa)} \ne 0,\\ 0, & \textit{otherwise},\end{cases}$$

$$C_\kappa = 1 + c_0 + c_0^{-2}\,\|c - \nabla\cdot b\|_{\mathrm{L}_\infty(\kappa)}^2, \quad D_\kappa = \delta_\kappa\|b\|_{\mathrm{L}_\infty(\kappa)}p/h_\kappa.$$

Theorem 3.5 is an extension of classical *a priori* error bounds for the streamline-diffusion discretisation of a first-order hyperbolic problem and a second-order elliptic transport-diffusion problem with a isotropic and constant; see, for example, [37], [54] (h-version) and [29] (hp-version), and [33] for a proof in the case of $\Gamma_N = \emptyset$.

4. Discontinuous piecewise polynomial approximation: the discontinuous Galerkin method

Discontinuous Galerkin finite element methods (DGFEMs, for short) date back to the early 1970s; they were simultaneously proposed by Reed & Hill [52] in 1973 for the numerical solution of the neutron transport equation and by Nitsche [45] in 1971 as a nonstandard scheme for the approximation of second-order elliptic equations. Since the early 1970s there has been extensive work on the development of these methods for a wide range of applications; for an excellent historical survey of DGFEMs up until 2000 we refer to the paper of Cockburn, Karniadakis and Shu in the volume [19].

One of the key advantages of the DGFEM in comparison with standard Galerkin finite element methods based on continuous piecewise polynomials, such as the streamline-diffusion finite method discussed in the previous section, is their high degree of locality: the computational stencil of the DGFEM remains very compact even as the degree of the approximating polynomial is increased. Hence, high-order adaptive hp- and spectral element approximations may be handled in a particularly flexible and simple manner. Indeed, hp-adaptive DGFEMs offer tremendous gains in terms of computational efficiency in comparison with standard mesh refinement algorithms which only incorporate local h-refinement with a given (fixed) polynomial degree. For discussions concerning various hp-refinement strategies see [1], [8], [32], [34], [53], [58]. A further attractive property of the discontinuous Galerkin finite element method for a transport-dominated diffusion problem is that, unlike its counterpart based on continuous piecewise polynomials, the method is stable even in the absence of streamline-diffusion stabilisation.

In this section, we survey *a priori* and *a posteriori* error bounds for discontinuous Galerkin finite element approximations of second-order partial differential equations with nonnegative characteristic form. We shall then show how the *a posteriori* error bound can be used to drive an hp-adaptive finite element algorithm. The presentation in this section is based on the paper [25].

We consider shape-regular meshes $\mathcal{T}_h = \{\kappa\}$ that partition the domain Ω into open element domains κ, with possible *hanging nodes*. We shall suppose that the mesh is 1-irregular in the sense that there is at most one hanging node per $(d-1)$-dimensional element-face, e.g. the barycenter of the face. We denote by h the piecewise constant mesh function with $h(x) \equiv h_\kappa = \mathrm{diam}(\kappa)$ when x is in element κ. Let each $\kappa \in \mathcal{T}_h$ be a smooth bijective image of a fixed master element $\hat{\kappa}$, that is, $\kappa = F_\kappa(\hat{\kappa})$ for all $\kappa \in \mathcal{T}_h$, where $\hat{\kappa}$ is either the open unit simplex $\hat{\kappa}_S = \{\hat{x} = (\hat{x}_1, \ldots, \hat{x}_d) \in \mathbb{R}^d : 0 < x_1 + \cdots + x_d < 1,\ x_i > 0,\ i = 1, \ldots, d\}$, or the open hypercube

$\hat{\kappa}_{\mathrm{C}} = (-1, 1)^d$ in $\mathbb{R}^d$. On $\hat{\kappa}$ we define spaces of polynomials of degree $p \geq 1$ as follows: $\mathcal{Q}_p = \mathrm{span}\{\hat{x}^\alpha : 0 \leq \alpha_i \leq p,\ 1 \leq i \leq d\}$, $\mathcal{P}_p = \mathrm{span}\{\hat{x}^\alpha : 0 \leq |\alpha| \leq p\}$. To each $\kappa \in \mathcal{T}_h$ we assign an integer $p_\kappa \geq 1$; collecting the p_κ and F_κ in the vectors $\boldsymbol{p} = \{p_\kappa : \kappa \in \mathcal{T}_h\}$ and $\boldsymbol{F} = \{F_\kappa : \kappa \in \mathcal{T}_h\}$, respectively, we introduce the finite element space

$$S^{\boldsymbol{p}}(\Omega, \mathcal{T}_h, \boldsymbol{F}) = \{u \in \mathrm{L}_2(\Omega) : u|_\kappa \circ F_\kappa \in \mathcal{Q}_{p_\kappa} \text{ if } F_\kappa^{-1}(\kappa) = \hat{\kappa}_{\mathrm{C}} \\ \text{and } u|_\kappa \circ F_\kappa \in \mathcal{P}_{p_\kappa} \text{ if } F_\kappa^{-1}(\kappa) = \hat{\kappa}_{\mathrm{S}};\ \kappa \in \mathcal{T}_h\}.$$

We assign to $\mathcal{T}_h$ the *broken Sobolev space* of composite order $\boldsymbol{s} = \{s_\kappa : \kappa \in \mathcal{T}_h\}$ defined by $\mathrm{H}^{\boldsymbol{s}}(\Omega, \mathcal{T}_h) = \{u \in \mathrm{L}_2(\Omega) : u|_\kappa \in \mathrm{H}^{s_\kappa}(\kappa) \text{ for all } \kappa \in \mathcal{T}_h\}$, equipped with the *broken Sobolev norm*

$$\|u\|_{\boldsymbol{s}, \mathcal{T}_h} = \Big(\sum_{\kappa \in \mathcal{T}_h} \|u\|^2_{\mathrm{H}^{s_\kappa}(\kappa)} \Big)^{1/2}.$$

When $s_\kappa = s$ for all $\kappa \in \mathcal{T}_h$, we write $\mathrm{H}^s(\Omega, \mathcal{T}_h)$ and $\|u\|_{s, \mathcal{T}_h}$.

An *interior face* of $\mathcal{T}_h$ is defined as the (non-empty) $(d-1)$-dimensional interior of $\partial\kappa_i \cap \partial\kappa_j$, where κ_i and κ_j are two adjacent elements of $\mathcal{T}_h$, not necessarily matching. A *boundary face* of $\mathcal{T}_h$ is defined as the (non-empty) $(d-1)$-dimensional interior of $\partial\kappa \cap \Gamma$, where κ is a boundary element of $\mathcal{T}_h$. We denote by Γ_{int} the union of all interior faces of $\mathcal{T}_h$. Given a face $e \subset \Gamma_{\mathrm{int}}$, shared by the two elements κ_i and κ_j, where the indices i and j satisfy $i > j$, we write ν_e to denote the (numbering-dependent) unit normal vector which points from κ_i to κ_j; on boundary faces we put $\nu_e = \nu$. Further, for $v \in \mathrm{H}^1(\Omega, \mathcal{T}_h)$ we define the jump of v across e and the mean value of v on e, respectively, by $[v] = v|_{\partial\kappa_i \cap e} - v|_{\partial\kappa_j \cap e}$ and $\langle v \rangle = \frac{1}{2}(v|_{\partial\kappa_i \cap e} + v|_{\partial\kappa_j \cap e})$. On a boundary face $e \subset \partial\kappa$, we set $[v] = v|_{\partial\kappa \cap e}$ and $\langle v \rangle = v|_{\partial\kappa \cap e}$. Finally, given a function $v \in \mathrm{H}^1(\Omega, \mathcal{T}_h)$ and an element $\kappa \in \mathcal{T}_h$, we denote by v_κ^+ (respectively, v_κ^-) the interior (respectively, exterior) trace of v defined on $\partial\kappa$ (respectively, $\partial\kappa \backslash \Gamma$). Since below it will always be clear from the context which element κ in the subdivision $\mathcal{T}_h$ the quantities v_κ^+ and v_κ^- correspond to, for the sake of notational simplicity we shall suppress the letter κ in the subscript and write, respectively, v^+ and v^- instead. Given that κ is an element in the subdivision $\mathcal{T}_h$, we denote by $\partial\kappa$ the union of $(d-1)$-dimensional open faces of κ. Let $x \in \partial\kappa$ and suppose that $\nu_\kappa(x)$ denotes the unit outward normal vector to $\partial\kappa$ at x. With these conventions, we define the inflow and outflow parts of $\partial\kappa$, respectively, by $\partial_-\kappa = \{x \in \partial\kappa : b(x) \cdot \nu_\kappa(x) < 0\}$, $\partial_+\kappa = \{x \in \partial\kappa : b(x) \cdot \nu_\kappa(x) \geq 0\}$.

For simplicity of presentation, we suppose that each entry of the matrix a is piecewise continuous on $\mathcal{T}_h$ and belongs to $S^{\boldsymbol{0}}(\Omega, \mathcal{T}_h, \boldsymbol{F})$. With minor changes only, our results can easily be extended to the case when each entry of $\sqrt{a}$ belongs to $S^{\boldsymbol{q}}(\Omega, \mathcal{T}_h, \boldsymbol{F})$, where the composite polynomial degree vector $\boldsymbol{q}$ has nonnegative entries; for more general a, see [23]. In the following, we write $\bar{a} = |\sqrt{a}\,|_2^2$, where $|\cdot|_2$ denotes the matrix norm subordinate to the l_2-vector norm on $\mathbb{R}^d$ and $\bar{a}_\kappa = \bar{a}|_\kappa$; by $\bar{a}_{\tilde{\kappa}}$ we denote the arithmetic mean of the values $\bar{a}_{\kappa'}$ over those elements κ' (including κ itself) that share a $(d-1)$-dimensional face with κ.

The hp-DGFEM approximation of (1), (3) is defined as follows: find u_{DG} in $S^{\boldsymbol{p}}(\Omega, \mathcal{T}_h, \boldsymbol{F})$ such that

$$B_{\mathrm{DG}}(u_{\mathrm{DG}}, v) = \ell_{\mathrm{DG}}(v) \quad \text{for all } v \in S^{\boldsymbol{p}}(\Omega, \mathcal{T}_h, \boldsymbol{F}). \tag{13}$$

Here, the bilinear form $B_{\mathrm{DG}}(\,\cdot\,,\cdot\,)$ is defined by

$$B_{\mathrm{DG}}(w, v) = B_a(w, v) + B_b(w, v) + \theta B_e(v, w) - B_e(w, v) + B_\sigma(w, v)\,,$$

where

$$\begin{aligned}
B_a(w, v) &= \sum_{\kappa\in\mathcal{T}_h} \int_\kappa a\nabla w \cdot \nabla v \,\mathrm{d}x,\\
B_b(w, v) &= \sum_{\kappa\in\mathcal{T}_h} \Big\{ -\int_\kappa (w\, b\cdot\nabla v - cwv)\,\mathrm{d}x + \int_{\partial_+\kappa} (b\cdot\nu_\kappa)\, w^+ v^+ \,\mathrm{d}s\\
&\qquad\qquad + \int_{\partial_-\kappa\setminus\Gamma} (b\cdot\nu_\kappa)\, w^- v^+ \,\mathrm{d}s\Big\},\\
B_e(w, v) &= \int_{\Gamma_{\mathrm{int}}\cup\Gamma_{\mathrm{D}}} \langle (a\nabla w)\cdot\nu_e\rangle [v]\,\mathrm{d}s,\\
B_\sigma(w, v) &= \int_{\Gamma_{\mathrm{int}}\cup\Gamma_{\mathrm{D}}} \sigma [w][v]\,\mathrm{d}s,
\end{aligned}$$

and the linear functional $\ell_{\mathrm{DG}}(\,\cdot\,)$ is given by

$$\begin{aligned}
\ell_{\mathrm{DG}}(v) = \sum_{\kappa\in\mathcal{T}_h} \Big\{ &\int_\kappa f v\,\mathrm{d}x - \int_{\partial_-\kappa\cap(\Gamma_{\mathrm{D}}\cup\Gamma_-)} (b\cdot\nu_\kappa)\, g_{\mathrm{D}}\, v^+ \,\mathrm{d}s\\
&+ \int_{\partial\kappa\cap\Gamma_{\mathrm{D}}} \theta\, g_{\mathrm{D}}((a\nabla v^+)\cdot\nu_\kappa)\,\mathrm{d}s + \int_{\partial\kappa\cap\Gamma_{\mathrm{N}}} g_{\mathrm{N}} v^+ \,\mathrm{d}s\\
&+ \int_{\partial\kappa\cap\Gamma_{\mathrm{D}}} \sigma g_{\mathrm{D}} v^+ \,\mathrm{d}s\Big\}.
\end{aligned}$$

Here, σ is defined by

$$\sigma|_e = C_\sigma \frac{\langle \bar{a} p^2\rangle}{\langle h\rangle} \quad \text{for } e \subset \Gamma_{\mathrm{int}}\cup\Gamma_{\mathrm{D}}, \tag{14}$$

where C_σ is a positive constant, called the *discontinuity-penalisation* parameter; cf. [30]. We shall adopt the convention that edges $e \subset \Gamma_{\mathrm{int}}\cup\Gamma_{\mathrm{D}}$ with $\sigma|_e = 0$ are omitted from the integrals appearing in the definition of $B_\sigma(w, v)$ and $\ell_{\mathrm{DG}}(v)$, although we shall not highlight this explicitly in our notation; the same convention is adopted in the case of integrals where the integrand contains the factor $1/\sigma$. Thus, in particular, the definition of the DG-norm, cf. (15) below, is meaningful even if $\sigma|_e$ happens to be equal to zero on certain edges $e \subset \Gamma_{\mathrm{int}}\cup\Gamma_{\mathrm{D}}$, given that such edges are understood to be excluded from the region of integration.

Selecting the parameter $\theta = 1$ gives rise to the so-called *Nonsymmetric Interior Penalty* (*NIP*) *method*, while setting $\theta = -1$ yields the *Symmetric Interior Penalty* (*SIP*) *scheme*; in the following we write SIP/NIP to denote the symmetric/nonsymmetric versions of the interior penalty method.

While a symmetric discretisation of a symmetric differential operator seems quite natural, the NIP scheme is often preferred, especially for transport-dominated problems where the underlying discretisation matrix is nonsymmetric anyway, as it is stable for any choice of the parameter $C_\sigma > 0$; see, for example, [2], [30], [51], and Theorem 4.1 below. On the other hand, the SIP scheme is only stable when $C_\sigma > 0$ is chosen sufficiently large. In terms of accuracy, both schemes converge at the optimal rate when the error is measured in terms of the DG-norm (cf. (15) below), but the lack of adjoint consistency (see, [2]) of the NIP method leads to suboptimal convergence of the error when measured in terms of the L_2 norm. In this case, the SIP scheme is still optimally convergent, while the NIP method is suboptimal by a full order; however, numerical experiments indicate that in practice the L_2 norm of the error arising from the NIP scheme converges to zero at the optimal rate when the polynomial degree p is odd, cf. [30]. Thereby, in practice the loss of optimality of the NIP scheme when the error is measured in terms of the L_2 norm only arises for even p. However, we showed in [25] that, for $p \geq 2$, the lack of adjoint consistency of the NIP scheme leads to an even more dramatic deterioration of its convergence rate when the error is measured in terms of a certain (linear) target functional $J(\cdot)$, such as $J: v \mapsto \int_\Omega v(x)\psi(x)\,\mathrm{d}x$, for example, where ψ is a given weight-function: for fixed p the error measured in terms of $J(\cdot)$ behaves like $\mathcal{O}(h^{2p})$ when the SIP scheme is employed, while for the NIP scheme, in general we only have the rate of convergence $\mathcal{O}(h^p)$ as h tends to zero. For related work on *a posteriori* error estimation for DGFEMs with interior penalty, see e.g. Becker *et al.* [9], [10] and Rivière & Wheeler [53]. For further perspectives on the construction and postprocessing of DGFEMs, see [12], [18].

Before embarking on the analysis of the discontinuous Galerkin method (13), we define the DG-norm $|||\cdot|||_{\mathrm{DG}}$ by

$$
\begin{aligned}
|||w|||_{\mathrm{DG}}^2 = \sum_{\kappa\in\mathcal{T}_h} \Big(& \|\nabla w\|_{a,\kappa}^2 + \|c_0 w\|_\kappa^2 + \frac{1}{2}\|w^+\|_{\partial_-\kappa\cap(\Gamma_{\mathrm{D}}\cup\Gamma_-)}^2 \\
& + \frac{1}{2}\|w^+\|_{\partial_+\kappa\cap\Gamma}^2 + \frac{1}{2}\|w^+ - w^-\|_{\partial_-\kappa\setminus\Gamma}^2 \Big) \\
& + \int_{\Gamma_{\mathrm{int}}\cup\Gamma_{\mathrm{D}}} \sigma[w]^2\,\mathrm{d}s + \int_{\Gamma_{\mathrm{int}}\cup\Gamma_{\mathrm{D}}} \frac{1}{\sigma}\langle (a\,\nabla w)\cdot \nu_e\rangle^2\,\mathrm{d}s,
\end{aligned}
\tag{15}
$$

where $\|\nabla w\|_{a,\kappa}^2 = (a\nabla w, \nabla w)_\kappa$, $\|\cdot\|_\tau$, $\tau \subset \partial\kappa$, denotes the (semi)norm induced by the (semi)inner-product $(v, w)_\tau = \int_\tau |b\cdot\nu_\kappa| v w\,\mathrm{d}s$, and c_0 is as defined in (4). The above definition of $|||\cdot|||_{\mathrm{DG}}$ represents a slight modification of the norm considered in [30]; for $a > 0$, $b \equiv 0$, (15) corresponds to the norm proposed by Baumann *et al.* [8], [47] and Baker *et al.* [5], cf. [51]. With this notation, we state the following coercivity result for the bilinear form $B_{\mathrm{DG}}(\cdot,\cdot)$ over $S^p(\Omega, \mathcal{T}_h, \boldsymbol{F}) \times S^p(\Omega, \mathcal{T}_h, \boldsymbol{F})$.

Theorem 4.1. *With σ defined as in* (14)*, there exists a positive constant C, which depends only on the dimension d and the shape-regularity of $\mathcal{T}_h$, such that*

$$B_{\mathrm{DG}}(v, v) \geq C|||v|||_{\mathrm{DG}}^2 \quad \textit{for all } v \in S^{\boldsymbol{p}}(\Omega, \mathcal{T}_h, \boldsymbol{F}),$$

provided that the constant C_σ arising in the definition of the discontinuity penalisation parameter σ is chosen so that $C_\sigma > 0$ arbitrary when $\theta = 1$, and $C_\sigma \geq C'_\sigma$ with C'_σ sufficiently large when $\theta \neq 1$.

This result is an extension of the coercivity result derived by Prudhomme *et al.* [51] with $b \equiv 0$; see also [30] for the proof in the case when $\theta = 1$. For the case of $a \equiv 0$, the connection of stabilisation to upwinding has been discussed in [16]. In particular, Theorem 4.1 implies that (13) has a unique solution for any $f \in \mathrm{L}_2(\Omega)$, any $g_{\mathrm{D}} \in \mathrm{L}_2(\Gamma_{\mathrm{D}})$ and any $g_{\mathrm{N}} \in \mathrm{L}_2(\Gamma_{\mathrm{N}})$. Theorem 4.1 also indicates that while the NIP scheme is coercive over $S^{\boldsymbol{p}}(\Omega, \mathcal{T}_h, \boldsymbol{F}) \times S^{\boldsymbol{p}}(\Omega, \mathcal{T}_h, \boldsymbol{F})$ for any choice of the constant $C_\sigma > 0$ arising in the definition of the discontinuity-penalisation parameter σ, the SIP scheme (corresponding to $\theta = -1$) is only coercive if C_σ is chosen sufficiently large; see [25] for details about the minimum size of C_σ.

Henceforth, we shall assume that the solution u to the boundary value problem (1), (3) is sufficiently smooth: namely, $u \in \mathrm{H}^2(\Omega, \mathcal{T}_h)$ and that u and $(a\nabla u) \cdot \nu_e$ are continuous across each face $e \subset \partial\kappa \backslash \Gamma$ that intersects the subdomain of ellipticity, $\Omega_a = \{x \in \overline{\Omega} : \boldsymbol{\zeta}^\top a(x)\boldsymbol{\zeta} > 0 \text{ for all } \boldsymbol{\zeta} \in \mathbb{R}^d\}$. If this smoothness requirement is violated, the discretisation method has to be modified accordingly, cf. [30]. We note that under these assumptions, the following Galerkin orthogonality property holds:

$$B_{\mathrm{DG}}(u - u_{\mathrm{DG}}, v) = 0 \quad \text{for all } v \in S^{\boldsymbol{p}}(\Omega, \mathcal{T}_h, \boldsymbol{F}). \tag{16}$$

We shall assume that $b \in [\mathrm{W}^1_\infty(\Omega)]^d$ is such that

$$b \cdot \nabla_{\mathcal{T}_h} v \in S^{\boldsymbol{p}}(\Omega, \mathcal{T}, \boldsymbol{F}) \quad \text{for all } v \in S^{\boldsymbol{p}}(\Omega, \mathcal{T}, \boldsymbol{F}). \tag{17}$$

Let us denote by Π_p the orthogonal projector in $\mathrm{L}_2(\Omega)$ onto the finite element space $S^{\boldsymbol{p}}(\Omega, \mathcal{T}, \boldsymbol{F})$. We remark that this choice of projector is essential in the following *a priori* error analysis, in order to ensure that $(u - \Pi_p u, b \cdot \nabla_{\mathcal{T}_h} v) = 0$ for all v in $S^{\boldsymbol{p}}(\Omega, \mathcal{T}, \boldsymbol{F})$. If, on the other hand, the scheme (13) is supplemented by streamline-diffusion stabilisation, then alternative choices of Π_p may be employed (see [29], [59], for example); in that case, hypothesis (17) is redundant. We now decompose the global error $u - u_{\mathrm{DG}}$ as

$$u - u_{\mathrm{DG}} = (u - \Pi_p u) + (\Pi_p u - u_{\mathrm{DG}}) \equiv \eta + \xi. \tag{18}$$

Lemma 4.2. *Assume that* (4) *and* (17) *hold and let $\beta_1|_\kappa = \|c/c_0^2\|_{\mathrm{L}_\infty(\kappa)}$; then there exists a positive constant C that depends only on d and the shape-regularity of $\mathcal{T}_h$*

such that the functions ξ and η defined by (18) *satisfy the following inequality*

$$|||\xi|||_{\mathrm{DG}}^2 \le C\Big\{ \sum_{\kappa\in\mathcal{T}_h} \big(\|\sqrt{a}\,\nabla\eta\|_\kappa^2 + \beta_1^2\|c_0\eta\|_\kappa^2 + \|\eta^+\|_{\partial_+\kappa\cap\Gamma}^2 + \|\eta^-\|_{\partial_-\kappa\setminus\Gamma}^2\big) + \int_{\Gamma_{\mathrm{int}}\cup\Gamma_{\mathrm{D}}} \frac{1}{\sigma}\langle(a\nabla\eta)\cdot\nu_e\rangle^2\,\mathrm{d}s + \int_{\Gamma_{\mathrm{int}}\cup\Gamma_{\mathrm{D}}} \sigma[\eta]^2\,\mathrm{d}s\Big\}.$$

The proof is given in [25]. We also need the following result concerning the approximation properties of the projector Π_p; for simplicity, we restrict ourselves to 1-irregular, shape-regular meshes consisting of affine equivalent d-parallelepiped elements (cf. [4], [30], and also [24] for similar results in augmented Sobolev spaces).

Lemma 4.3. *Suppose that $\kappa \in \mathcal{T}_h$ is a d-parallelepiped of diameter h_κ and that $u|_\kappa \in \mathrm{H}^{k_\kappa}(\kappa)$, $k_\kappa \ge 0$, for $\kappa \in \mathcal{T}_h$. Then, the following approximation results hold:*

$$\|u-\Pi_p u\|_{\mathrm{L}_2(\kappa)} \le C\frac{h_\kappa^{s_\kappa}}{p_\kappa^{k_\kappa}}\|u\|_{\mathrm{H}^{k_\kappa}(\kappa)}, \quad \|u-\Pi_p u\|_{\mathrm{L}_2(\partial\kappa)} \le C\frac{h_\kappa^{s_\kappa-1/2}}{p_\kappa^{k_\kappa-1/2}}\|u\|_{\mathrm{H}^{k_\kappa}(\kappa)},$$

$$|u-\Pi_p u|_{\mathrm{H}^1(\kappa)} \le C\frac{h_\kappa^{s_\kappa-1}}{p_\kappa^{k_\kappa-3/2}}\|u\|_{\mathrm{H}^{k_\kappa}(\kappa)}, \quad |u-\Pi_p u|_{\mathrm{H}^1(\partial\kappa)} \le C\frac{h_\kappa^{s_\kappa-3/2}}{p_\kappa^{k_\kappa-5/2}}\|u\|_{\mathrm{H}^{k_\kappa}(\kappa)},$$

where $1 \le s_\kappa \le \min(p_\kappa+1, k_\kappa)$ and C is a constant independent of u, h_κ and p_κ, but dependent on the dimension d and the shape-regularity of $\mathcal{T}_h$.

For the rest of this section, we assume that the polynomial degree vector $\boldsymbol{p}$, with $p_\kappa \ge 1$, $\kappa \in \mathcal{T}_h$, has *bounded local variation*; *i.e.*, there exists a constant $\rho \ge 1$ such that, for any pair of elements κ and κ' which share a $(d-1)$-dimensional face,

$$\rho^{-1} \le p_\kappa/p_{\kappa'} \le \rho. \tag{19}$$

On noting that $\eta = u - \Pi_p u$ and combining Lemmas 4.2 and 4.3, we deduce that

$$|||\xi|||_{\mathrm{DG}}^2 \le C\sum_{\kappa\in\mathcal{T}_h}\left(\alpha\frac{h_\kappa^{2(s_\kappa-1)}}{p_\kappa^{2(k_\kappa-3/2)}} + \beta_2\frac{h_\kappa^{2s_\kappa}}{p_\kappa^{2k_\kappa}} + \gamma\frac{h_\kappa^{2(s_\kappa-1/2)}}{p_\kappa^{2(k_\kappa-1/2)}}\right)\|u\|_{\mathrm{H}^{k_\kappa}(\kappa)}^2,$$

where $\alpha|_\kappa = \bar{a}_{\tilde{\kappa}}$, $\beta_2|_\kappa = (\beta_1|_\kappa)^2\,\|c_0\|_{\mathrm{L}_\infty(\kappa)}^2$, $(\beta_1|_\kappa = \|c/(c_0)^2\|_{\mathrm{L}_\infty(\kappa)}$, cf. Lemma 4.2), $\gamma|_\kappa = \|b\|_{\mathrm{L}_\infty(\kappa)}$ and C is a positive constant that depends only on d, the parameter ρ in (19) and the shape-regularity of $\mathcal{T}_h$. The DG-norm $|||\eta|||_{\mathrm{DG}}$ of $\eta = u - \Pi_p u$ can be estimated directly using Lemma 4.3 to show that a bound analogous to that on $|||\xi|||_{\mathrm{DG}}$ above holds. Hence, a bound on the discretisation error $u - u_{\mathrm{DG}} = \xi + \eta$ in the DG-norm $|||\cdot|||_{\mathrm{DG}}$ is obtained via the triangle inequality (see [30] for details).

Very often in practice the objective of the computation is not the approximation of the analytical solution u in a given norm, but controlling the error in an output/target-functional $J(\cdot)$ of the solution. Relevant examples of output functionals include the lift and drag coefficients of a body immersed into a viscous fluid, the local mean value

of the field, or its flux through the outflow boundary of the computational domain. Here we give a brief survey of *a posteriori* and *a priori* error bounds for general linear target functionals $J(\cdot)$ of the solution; for related work, we refer to [11], [26], [28], [31], [32], [34], [42], [58], [59], for example, and to the recent monograph of Bangerth & Rannacher [6]. We shall confine ourselves to Type I (dual-weighted) *a posteriori* bounds; the computationally simpler, but cruder, Type II error bounds will not be discussed here (see Giles & Süli [22]).

Type I *a posteriori* error analysis. We proceed as in [34], [58] and begin by considering the following *dual* or *adjoint* problem: find $z \in \mathrm{H}^2(\Omega, \mathcal{T}_h)$ such that

$$B_{\mathrm{DG}}(w, z) = J(w) \quad \text{for all } w \in \mathrm{H}^2(\Omega, \mathcal{T}_h). \tag{20}$$

Let us assume that (20) possesses a unique solution. Clearly, the validity of this assumption depends, *inter alia*, on the choice of the linear functional under consideration. We shall return to this issue below; see also [25], [28], [34].

For a given linear functional $J(\cdot)$ the *a posteriori* error bound will be expressed in terms of the finite element residual R_{int} defined on $\kappa \in \mathcal{T}_h$ by $R_{\mathrm{int}}|_\kappa = (f - \mathcal{L}u_{\mathrm{DG}})|_\kappa$, which measures the extent to which u_{DG} fails to satisfy the differential equation on the union of the elements κ in the mesh $\mathcal{T}_h$; thus we refer to R_{int} as the *internal residual*. Also, since u_{DG} only satisfies the boundary conditions approximately, the differences $g_{\mathrm{D}} - u_{\mathrm{DG}}$ and $g_{\mathrm{N}} - (a\nabla u_{\mathrm{DG}}) \cdot \nu$ are not necessarily zero on $\Gamma_{\mathrm{D}} \cup \Gamma_-$ and Γ_{N}, respectively; thus we define the *boundary residuals* R_{D} and R_{N} by $R_{\mathrm{D}}|_{\partial\kappa\cap(\Gamma_{\mathrm{D}}\cup\Gamma_-)} = (g_{\mathrm{D}} - u^+_{\mathrm{DG}})|_{\partial\kappa\cap(\Gamma_{\mathrm{D}}\cup\Gamma_-)}$ and $R_{\mathrm{N}}|_{\partial\kappa\cap\Gamma_{\mathrm{N}}} = (g_{\mathrm{N}} - (a\nabla u^+_{\mathrm{DG}}) \cdot \nu)|_{\partial\kappa\cap\Gamma_{\mathrm{N}}}$, respectively. By using the divergence theorem, the Galerkin orthogonality property (16) may be rewritten as follows:

$$\begin{aligned}
0 = B_{\mathrm{DG}}(u - u_{\mathrm{DG}}, v) &= \ell_{\mathrm{DG}}(v) - B_{\mathrm{DG}}(u_{\mathrm{DG}}, v) \\
&= \sum_{\kappa\in\mathcal{T}_h} \Bigg\{ \int_\kappa R_{\mathrm{int}} v \,\mathrm{d}x - \int_{\partial_-\kappa\cap\Gamma} (b\cdot\nu_\kappa)\, R_{\mathrm{D}} v^+ \,\mathrm{d}s \\
&\qquad + \int_{\partial_-\kappa\setminus\Gamma} (b\cdot\nu_\kappa)\,[u_{\mathrm{DG}}]v^+ \,\mathrm{d}s + \int_{\partial\kappa\cap\Gamma_{\mathrm{D}}} \theta\, R_{\mathrm{D}}((a\nabla v^+)\cdot\nu_\kappa)\,\mathrm{d}s \\
&\qquad + \int_{\partial\kappa\cap\Gamma_{\mathrm{D}}} \sigma R_{\mathrm{D}} v^+ \,\mathrm{d}s + \int_{\partial\kappa\cap\Gamma_{\mathrm{N}}} R_{\mathrm{N}} v^+ \,\mathrm{d}s \\
&\qquad - \int_{\partial\kappa\setminus\Gamma} \left(\frac{\theta}{2}\,[u_{\mathrm{DG}}](a\nabla v^+)\cdot\nu_\kappa + \frac{1}{2}[(a\nabla u_{\mathrm{DG}})\cdot\nu_\kappa]v^+ \right) \mathrm{d}s \\
&\qquad - \int_{\partial\kappa\setminus\Gamma} \sigma [u_{\mathrm{DG}}] v^+ \,\mathrm{d}s \Bigg\}
\end{aligned} \tag{21}$$

for all $v \in S^{\boldsymbol{p}}(\Omega, \mathcal{T}_h, \boldsymbol{F})$. The starting point is the following result from [25].

Theorem 4.4. *Let u and u_{DG} denote the solutions of* (1), (3) *and* (13), *respectively, and suppose that the dual solution z is defined by* (20). *Then, the following error*

representation formula holds:

$$J(u) - J(u_{\mathrm{DG}}) = \mathcal{E}_{\Omega}(u_{\mathrm{DG}}, h, p, z - z_{h,p}) \equiv \sum_{\kappa \in \mathcal{T}_h} \eta_{\kappa}, \tag{22}$$

where

$$\begin{aligned}
\eta_{\kappa} = & \int_{\kappa} R_{\mathrm{int}}(z - z_{h,p})\,\mathrm{d}x - \int_{\partial_{-}\kappa \cap \Gamma} (b \cdot \nu_{\kappa})\, R_{\mathrm{D}}(z - z_{h,p})^{+}\,\mathrm{d}s \\
& + \int_{\partial_{-}\kappa \setminus \Gamma} (b \cdot \nu_{\kappa})\, [u_{\mathrm{DG}}](z - z_{h,p})^{+}\,\mathrm{d}s \\
& + \int_{\partial\kappa \cap \Gamma_{\mathrm{D}}} \theta\, R_{\mathrm{D}}((a\nabla(z - z_{h,p})^{+}) \cdot \nu_{\kappa})\,\mathrm{d}s \\
& + \int_{\partial\kappa \cap \Gamma_{\mathrm{D}}} \sigma\, R_{\mathrm{D}}(z - z_{h,p})^{+}\,\mathrm{d}s + \int_{\partial\kappa \cap \Gamma_{\mathrm{N}}} R_{\mathrm{N}}(z - z_{h,p})^{+}\,\mathrm{d}s \\
& - \int_{\partial\kappa \setminus \Gamma} \left\{ \frac{\theta}{2}\, [u_{\mathrm{DG}}](a\nabla(z - z_{h,p})^{+}) \cdot \nu_{\kappa} + \frac{1}{2}[(a\nabla u_{\mathrm{DG}}) \cdot \nu_{\kappa}](z - z_{h,p})^{+} \right\} \mathrm{d}s \\
& - \int_{\partial\kappa \setminus \Gamma} \sigma [u_{\mathrm{DG}}](z - z_{h,p})^{+}\,\mathrm{d}s
\end{aligned} \tag{23}$$

for all $z_{h,p} \in S^{\boldsymbol{p}}(\Omega, \mathcal{T}_h, \boldsymbol{F})$.

Proof. On choosing $w = u - u_{\mathrm{DG}}$ in (20) and recalling the linearity of $J(\cdot)$ and the Galerkin orthogonality property (21), we deduce that

$$\begin{aligned}
J(u) - J(u_{\mathrm{DG}}) &= J(u - u_{\mathrm{DG}}) = B_{\mathrm{DG}}(u - u_{\mathrm{DG}}, z) \\
&= B_{\mathrm{DG}}(u - u_{\mathrm{DG}}, z - z_{h,p}),
\end{aligned} \tag{24}$$

and hence (22), with η_{κ} defined by (23), using the definitions of the residuals. □

Corollary 4.5. *Under the assumptions of Theorem* 4.4*, and with* η_{κ} *defined as in* (23)*, the following Type I* a posteriori *error bound holds:*

$$|J(u) - J(u_{\mathrm{DG}})| \leq \mathcal{E}_{|\Omega|}(u_{\mathrm{DG}}, h, p, z - z_{h,p}) \equiv \sum_{\kappa \in \mathcal{T}_h} |\eta_{\kappa}|\,. \tag{25}$$

As discussed in [6], [11], [26], [58], the local *weighting terms* involving the difference between the dual solution z and its projection/interpolant $z_{h,p}$ onto $S^{\boldsymbol{p}}(\Omega, \mathcal{T}_h, \boldsymbol{F})$ appearing in the Type I bound (25) contain useful information concerning the global transport of the error. Therefore, we shall retain the weighting terms involving the (unknown) dual solution z in our bound and approximate them numerically, — instead of eliminating z, as one would in the derivation of a structurally simpler, but cruder, Type II *a posteriori* bound. However, before proceeding any further, we need to consider more carefully the dual problem defined by (20). Let us suppose, for example,

that the aim of the computation is to approximate the (weighted) mean value of the solution u; *i.e.*, $J(\cdot) \equiv M_\psi(\cdot)$, where $M_\psi(w) = \int_\Omega w\psi \, \mathrm{d}x$ and $\psi \in \mathrm{L}_2(\Omega)$. When $\theta = -1$, performing integration by parts, we find that the corresponding dual solution $z = z^{\mathrm{SIP}}$ is the solution of the following mesh-dependent problem: find z such that

$$\begin{aligned} \mathcal{L}^* z \equiv -\nabla \cdot (a\nabla z) - b \cdot \nabla z + cz = \psi \quad & \text{in } \Omega, \\ z = 0 \quad & \text{on } \Gamma_{\mathrm{D}} \cup \Gamma_+, \\ (b \cdot \nu)z + (a\nabla z) \cdot \nu = 0 \quad & \text{on } \Gamma_{\mathrm{N}}. \end{aligned}$$

Thus, for $\theta = -1$ the dual problem is well-posed for this choice of target functional. We remark that, since in this case the dual problem formed by transposing the arguments in the bilinear form $B_{\mathrm{DG}}(\cdot,\cdot) = B^{\mathrm{SIP}}_{\mathrm{DG}}(\cdot,\cdot)$ involves the formal adjoint of the partial differential operator $\mathcal{L}$, $B^{\mathrm{SIP}}_{\mathrm{DG}}(\cdot,\cdot)$ is said to be *adjoint consistent*, cf. Arnold *et al.* [2]; in particular, when $\theta = -1$ and the primal and dual solutions are sufficiently smooth, the error in the functional will be seen to exhibit an optimal order of convergence. As we shall explain below by means of *a priori* error analysis, the situation is dramatically different when $\theta \neq -1$: then, the bilinear form $B_{\mathrm{DG}}(\cdot,\cdot)$ is *not* adjoint consistent; this, in turn, leads to degradation of the convergence rate of the error in the computed functional $J(\cdot)$ as the finite element space $S^{\boldsymbol{p}}(\Omega, \mathcal{T}_h, \boldsymbol{F})$ is enriched (by reducing h or by increasing the polynomial degree vector $\boldsymbol{p}$). Once again, we refer to [25] for technical details.

***A priori* error bounds.** We continue to use the superscripts SIP and NIP to distinguish between the two methods and write $B^{\mathrm{SIP}}_{\mathrm{DG}}(\cdot,\cdot) \equiv B_{\mathrm{DG}}(\cdot,\cdot)$ when $\theta = -1$ and $B^{\mathrm{NIP}}_{\mathrm{DG}}(\cdot,\cdot) \equiv B_{\mathrm{DG}}(\cdot,\cdot)$ when $\theta = 1$. The corresponding numerical solutions $u^{\mathrm{SIP}}_{\mathrm{DG}}$ and $u^{\mathrm{NIP}}_{\mathrm{DG}}$ satisfy the following problems: find $u^{\mathrm{SIP}}_{\mathrm{DG}}$ in $S^{\boldsymbol{p}}(\Omega, \mathcal{T}_h, \boldsymbol{F})$ such that

$$B^{\mathrm{SIP}}_{\mathrm{DG}}(u^{\mathrm{SIP}}_{\mathrm{DG}}, v) = \ell_{\mathrm{DG}}(v) \quad \text{for all } v \in S^{\boldsymbol{p}}(\Omega, \mathcal{T}_h, \boldsymbol{F});$$

and find $u^{\mathrm{NIP}}_{\mathrm{DG}}$ in $S^{\boldsymbol{p}}(\Omega, \mathcal{T}_h, \boldsymbol{F})$ such that

$$B^{\mathrm{NIP}}_{\mathrm{DG}}(u^{\mathrm{NIP}}_{\mathrm{DG}}, v) = \ell_{\mathrm{DG}}(v) \quad \text{for all } v \in S^{\boldsymbol{p}}(\Omega, \mathcal{T}_h, \boldsymbol{F}),$$

respectively. The starting point for the *a priori* error analysis is the identity (24) in the proof of Theorem 4.4. Again, using the above notation, we see that

$$J(u) - J(u^{\mathrm{SIP}}_{\mathrm{DG}}) = B^{\mathrm{SIP}}_{\mathrm{DG}}(u - u^{\mathrm{SIP}}_{\mathrm{DG}}, z^{\mathrm{SIP}} - z_{h,p})$$

when the SIP scheme is employed, while for the NIP scheme, we have

$$J(u) - J(u^{\mathrm{NIP}}_{\mathrm{DG}}) = B^{\mathrm{NIP}}_{\mathrm{DG}}(u - u^{\mathrm{NIP}}_{\mathrm{DG}}, z^{\mathrm{NIP}} - z_{h,p})$$

for all $z_{h,p}$ in $S^{\boldsymbol{p}}(\Omega, \mathcal{T}_h, \boldsymbol{F})$. Here, z^{SIP} and z^{NIP} are the analytical solutions to the following dual problems: find $z^{\mathrm{SIP}} \in \mathrm{H}^2(\Omega, \mathcal{T}_h)$ such that

$$B^{\mathrm{SIP}}_{\mathrm{DG}}(w, z^{\mathrm{SIP}}) = J(w) \quad \text{for all } w \in \mathrm{H}^2(\Omega, \mathcal{T}_h);$$

and find $z^{\text{NIP}} \in \mathrm{H}^2(\Omega, \mathcal{T}_h)$ such that

$$B_{\text{DG}}^{\text{NIP}}(w, z^{\text{NIP}}) = J(w) \quad \text{for all } w \in \mathrm{H}^2(\Omega, \mathcal{T}_h).$$

Hence, for all $z_{h,p} \in S^{\boldsymbol{p}}(\Omega, \mathcal{T}_h, \boldsymbol{F})$,

$$J(u) - J(u_{\text{DG}}) = B_{\text{DG}}(u - u_{\text{DG}}, z^{\text{SIP}} - z_{h,p}) - (1+\theta) B_e(z^{\text{SIP}}, u - u_{\text{DG}}^{\text{NIP}}), \quad (26)$$

where u_{DG} is either $u_{\text{DG}}^{\text{SIP}}$ or $u_{\text{DG}}^{\text{NIP}}$, depending on whether $\theta = -1$ or $\theta = 1$, respectively. In particular, the second term on the right-hand side of (26) is absent for the SIP scheme, i.e. when $\theta = -1$, but it is present when the NIP scheme is employed, *i.e.*, when $\theta = 1$. Since this second term is of lower order than the first term in (26), it will lead to suboptimal rates of convergence as the finite element space $S^{\boldsymbol{p}}(\Omega, \mathcal{T}_h, \boldsymbol{F})$ is enriched in the case of $\theta \neq -1$.

Theorem 4.6. *Let $\Omega \subset \mathbb{R}^d$ be a bounded polyhedral domain, $\mathcal{T}_h = \{\kappa\}$ a shape-regular subdivision of Ω into d-parallelepipeds and $\boldsymbol{p}$ a polynomial degree vector of bounded local variation. Let* (4) *and* (17) *hold, let the entries of a be piecewise constant on $\mathcal{T}_h$, and $u|_\kappa \in \mathrm{H}^{k_\kappa}(\kappa)$, $k_\kappa \geq 2$, for $\kappa \in \mathcal{T}_h$, $z^{\text{SIP}}|_\kappa \in \mathrm{H}^{l_\kappa}(\kappa)$, $l_\kappa \geq 2$, for $\kappa \in \mathcal{T}_h$; then, the solution $u_{\text{DG}} \in S^{\boldsymbol{p}}(\Omega, \mathcal{T}_h, \boldsymbol{F})$ of* (13) *satisfies the error bound*

$$|J(u) - J(u_{\text{DG}})|^2 \leq C \left\{ \sum_{\kappa \in \mathcal{T}_h} \left(\alpha \frac{h_\kappa^{2(s_\kappa - 1)}}{p_\kappa^{2(k_\kappa - 3/2)}} + \beta_3 \frac{h_\kappa^{2s_\kappa}}{p_\kappa^{2k_\kappa}} + \gamma \frac{h_\kappa^{2(s_\kappa - 1/2)}}{p_\kappa^{2(k_\kappa - 1/2)}} \right) \|u\|^2_{\mathrm{H}^{k_\kappa}(\kappa)} \right\}$$

$$\times \left\{ \sum_{\kappa \in \mathcal{T}_h} \left(\alpha \frac{h_\kappa^{2(t_\kappa - 1)}}{p_\kappa^{2(l_\kappa - 3/2)}} + \beta_4 \frac{h_\kappa^{2t_\kappa}}{p_\kappa^{2l_\kappa}} + \gamma \frac{h_\kappa^{2(t_\kappa - 1/2)}}{p_\kappa^{2(l_\kappa - 1)}} \right) \|z^{\text{SIP}}\|^2_{\mathrm{H}^{l_\kappa}(\kappa)} + (1+\theta) \|z^{\text{SIP}}\|^2_{2,\mathcal{T}_h} \right\}$$

for $1 \leq s_\kappa \leq \min(p_\kappa + 1, k_\kappa)$, $1 \leq t_\kappa \leq \min(p_\kappa + 1, l_\kappa)$, $p_\kappa \geq 1$, $\kappa \in \mathcal{T}_h$, where $\alpha|_\kappa = \bar{a}_{\tilde{\kappa}}$, $\beta_3|_\kappa = (1 + (\beta_1|_\kappa)^2)\|c_0\|^2_{\mathrm{L}_\infty(\kappa)}$, $(\beta_1|_\kappa = \|c(x)/(c_0(x))^2\|_{\mathrm{L}_\infty(\kappa)})$, $\beta_4|_\kappa = \|(c + \nabla \cdot b)/c_0\|^2_{\mathrm{L}_\infty(\kappa)}$, $\gamma|_\kappa = \|b\|_{\mathrm{L}_\infty(\kappa)}$ and C is a constant depending on the dimension d, the parameter ρ from (19) *and the shape-regularity of $\mathcal{T}_h$.*

If we assume uniform orders $p_\kappa = p$, $s_\kappa = s$, $t_\kappa = t$, $k_\kappa = k$, $l_\kappa = l$, where s, t, k and l are positive integers, and $h = \max_{\kappa \in \mathcal{T}_h} h_\kappa$, then, in the diffusion-dominated case (viz. $b \approx 0$), Theorem 4.6, with $\theta = -1$ implies that for the SIP scheme

$$|J(u) - J(u_{\text{DG}})| \leq C\, (h^{s+t-2}/p^{k+l-2})\, p\, \|u\|_{\mathrm{H}^k(\Omega)} \|z^{\text{SIP}}\|_{\mathrm{H}^l(\Omega)}, \quad (27)$$

where $1 \leq s \leq \min(p+1, k)$ and $1 \leq t \leq \min(p+1, l)$. This error bound is optimal with respect to h and suboptimal in p by a full order. We note, however, that 'order-doubling' of the rate of convergence in $|J(u) - J(u_{\text{DG}})|$ observed when the SIP scheme is employed, as expressed by (27), is lost when the NIP method is used. In the hyperbolic case ($a \equiv 0$), the bound in Theorem 4.6 becomes

$$|J(u) - J(u_{\text{DG}})| \quad \leq \quad C\, (h^{s+t-1}/p^{k+l-1})\, p^{1/2}\, \|u\|_{\mathrm{H}^k(\Omega)} \|z^{\text{SIP}}\|_{\mathrm{H}^l(\Omega)}.$$

This error bound is optimal in h and suboptimal in p by $p^{1/2}$ (cf. also [34]).

Adaptive algorithm. In the light of Theorem 4.6, we now confine ourselves to the SIP scheme ($\theta = -1$). For a user-defined tolerance TOL, we consider the problem of designing an hp-finite element space $S^{\boldsymbol{p}}(\Omega, \mathcal{T}_h, \boldsymbol{F})$ such that the inequality $|J(u) - J(u_{\mathrm{DG}})| \leq \mathtt{TOL}$ holds, subject to the constraint that the number of degrees of freedom in $S^{\boldsymbol{p}}(\Omega, \mathcal{T}_h, \boldsymbol{F})$ is minimized. Following [34], we use the *a posteriori* error bound (25) with z replaced by a discontinuous Galerkin approximation $\hat{z}$ computed on the same mesh $\mathcal{T}_h$ as for the primal solution u_{DG}, but with a higher degree polynomial, *i.e.*, $\hat{z} \in S^{\hat{\boldsymbol{p}}}(\Omega, \mathcal{T}_h, \boldsymbol{F})$, $\hat{\boldsymbol{p}} = \boldsymbol{p} + \boldsymbol{p}_{\mathrm{inc}}$; in our experiments we set $\boldsymbol{p}_{\mathrm{inc}} = \mathbf{1}$, cf. [26], [32], [58]. Thereby, in practice we enforce the stopping criterion

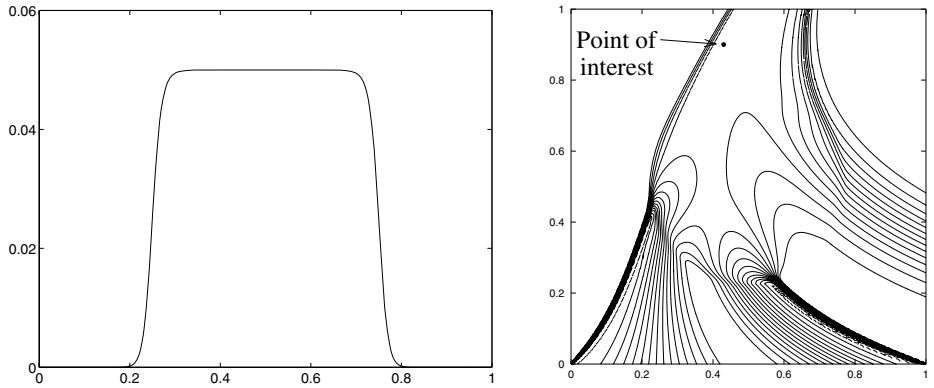
$$\hat{\mathcal{E}}_{|\Omega|} \equiv \mathcal{E}_{|\Omega|}(u_{\mathrm{DG}}, h, p, \hat{z} - z_{h,p}) \leq \mathtt{TOL}\,. \tag{28}$$

If (28) is not satisfied, then the elements are marked for refinement/derefinement according to the size of the (approximate) error indicators $|\hat{\eta}_\kappa|$; these are defined analogously to $|\eta_\kappa|$ in (23) with z replaced by $\hat{z}$. In our experiments we use the fixed fraction mesh refinement algorithm, with refinement and derefinement fractions set to 20% and 10%, respectively.

Once an element $\kappa \in \mathcal{T}_h$ has been flagged for refinement or derefinement, a decision must be made whether the local mesh size h_κ or the local degree p_κ of the approximating polynomial should be altered. The choice to perform either h-refinement/derefinement or p-refinement/derefinement is based on the local smoothness of the primal and dual solutions u and z, respectively; cf. [32], [34]. Let us first consider the case when an element has been flagged for *refinement*. If u or z are locally smooth, then p-refinement will be more effective than h-refinement, since the error will be expected to decay quickly within the current element κ as p_κ is increased. On the other hand, if both u and z have low regularity within the element κ, then h-refinement will be performed. To ensure that the desired level of accuracy is achieved efficiently, in [34] an automatic procedure was developed for deciding when to h- or p-refine, based on the smoothness-estimation strategy proposed by Ainsworth & Senior [1]. For a review of various hp-adaptive strategies as well as descriptions of new algorithms based on Sobolev index estimation *via* local Legendre expansions, we refer to [31], [32]. If an element has been flagged for *derefinement*, then the strategy implemented here is to coarsen the mesh in low-error-regions where either the primal or dual solutions u and z, respectively, are smooth and decrease the degree of the approximating polynomial in low-error-regions when both u and z are insufficiently regular, cf. [34].

Numerical experiments. We explore the performance of the hp-adaptive strategy outlined above for the symmetric version of the interior penalty method, applied to a mixed hyperbolic-elliptic problem with discontinuous boundary data (cf. [25]). We let $a = \varepsilon(x)I$, where $\varepsilon = \frac{1}{2}\delta(1 - \tanh((r - 1/4)(r + 1/4)/\gamma))$, $r^2 = (x - 1/2)^2 + (y - 1/2)^2$ and $\delta \geq 0$ and $\gamma > 0$ are constants. Let $b = (2y^2 - 4x + 1, 1 + y)$, $c = -\nabla \cdot b$ and $f = 0$. With $\delta > 0$ and $0 < \gamma \ll 1$, the diffusion parameter ε

is approximately equal to δ in the circular region defined by $r < 1/4$, where the underlying partial differential equation is uniformly elliptic. In this example, we set $\delta = 0.05$ and $\gamma = 0.01$; a cross-section of ε along $0 \le x \le 1$, $y = 1/2$ is shown in Figure 1. As the value of r is increased beyond $1/4$, the function ε rapidly decreases through a layer of width $\mathcal{O}(\gamma)$; for example, when $r > 0.336$ we have $\varepsilon < 10^{-15}$, so from the computational point of view ε is zero to within rounding error; in this



(a) (b)

Figure 1. (a) Profile of ε along $y = 0.5$, $0 \le x \le 1$; (b) DGFEM approximation to the primal problem on a 129×129 mesh with piecewise bilinear elements ($\boldsymbol{p} = \mathbf{1}$); from [25].

region, the partial differential equation undergoes a change of type becoming, in effect, hyperbolic. Thus we shall refer to the part of Ω with $r > 1/4 + \mathcal{O}(\gamma)$ as the *hyperbolic region*, while the set of points in Ω with $r \le 1/4$ will be called the *elliptic region*; of course, strictly speaking, the partial differential equation is elliptic in the whole of $\overline{\Omega}$. The characteristics associated with the hyperbolic part of the operator enter the computational domain Ω from three sides of Γ, namely through the vertical edges placed along $x = 0$ and $x = 1$ and the horizontal edge along $y = 0$; the characteristics exit Ω through the horizontal edge along $y = 1$. On the union of these three faces we prescribe the following boundary condition:

$$u(x, y) = \begin{cases} 1 & \text{for } x = 0 \ , \ 0 < y \le 1, \\ \sin^2(\pi x) & \text{for } 0 \le x \le 1 \ , \ y = 0, \\ \mathrm{e}^{-50y^4} & \text{for } x = 1 \, , \ 0 < y \le 1. \end{cases}$$

Figure 1 shows the numerical approximation to (1) using the SIP method on a uniform 129×129 uniform square mesh with $\boldsymbol{p} = \mathbf{1}$. Let us suppose that the objective of the computation is to calculate the value of the analytical solution u at a certain point of interest, $x = (0.43, 0.9)$, *i.e.*, $J(u) = u(0.43, 0.9)$; cf. Figure 1. The true value of the functional is given by $J(u) = 0.704611313375$.

In Table 1 we show the performance of our adaptive finite element algorithm using hp-refinement. Clearly, the computed Type I *a posteriori* error bound (25) is very sharp in the sense that it overestimates the true error in the computed functional by a factor of about 1–8 only, and by a factor of only 3.34 on average on the meshes that arise in the course of our adaptive hp-refinement.

Table 1. History of the adaptive hp-refinement. The effectivity index is defined as the ratio of the *a posteriori* error bound $\sum_\kappa |\hat{\eta}_\kappa|$ and the error $|J(u) - J(u_{\mathrm{DG}})|$; from [25].

Nodes	Elements	Degrees of freedom	$\lvert J(u) - J(u_{\mathrm{DG}})\rvert$	$\sum_\kappa \lvert\hat{\eta}_\kappa\rvert$	Effectivity index
81	64	576	1.924e-02	3.330e-02	1.73
99	76	740	1.056e-02	1.085e-02	1.03
162	130	1451	1.006e-02	2.290e-02	2.28
241	193	2483	7.400e-04	2.385e-03	3.22
302	244	3776	3.760e-05	2.754e-04	7.32
323	262	4777	1.270e-05	1.026e-04	8.08
396	325	6916	9.896e-06	2.245e-05	2.27
487	403	9941	1.224e-06	6.466e-06	5.28
577	481	13528	4.656e-07	1.163e-06	2.50
713	601	19855	2.449e-07	2.582e-07	1.05
960	820	31019	1.574e-08	3.202e-08	2.03
1313	1132	47406	6.531e-10	2.154e-09	3.30

Figure 2 shows $|J(u) - J(u_{\mathrm{DG}})|$, using both h- and hp-refinement, against the square-root of the number of degrees of freedom on a linear-log scale. After the initial transient, the error in the computed functional using hp-refinement is seen to become (on average) a straight line, which indicates exponential convergence of $J(u_{\mathrm{DG}})$ to $J(u)$; this occurs since z^{SIP} is a real analytic function in the regions of the computational domain where u is not smooth and *vice versa*. Figure 2 also demonstrates the superiority of the adaptive hp-refinement strategy over the standard adaptive h-refinement algorithm when $\mathtt{TOL} \lessapprox 10^{-3}$.

On the final mesh the error between $J(u)$ and $J(u_{\mathrm{DG}})$ using hp-refinement is over 4 orders of magnitude smaller than the corresponding quantity when h-refinement is used alone.

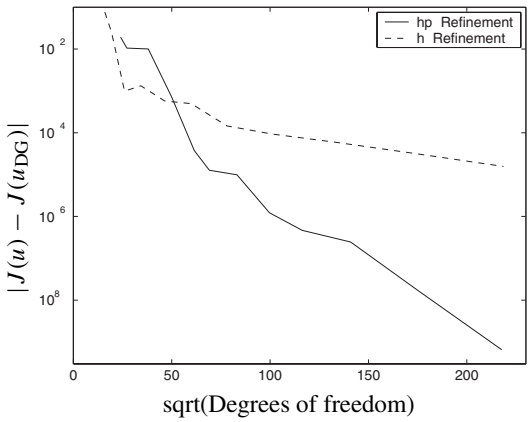


Figure 2. Comparison between h- and hp-adaptive mesh refinement; from [25].

Figure 3 depicts the primal mesh after 11 adaptive mesh refinement steps. We display the h-mesh alone, as well as the corresponding distribution of the polynomial degree on this mesh and the percentage of elements with that degree. We see that some h-refinement of the primal mesh has taken place in the region of the computational

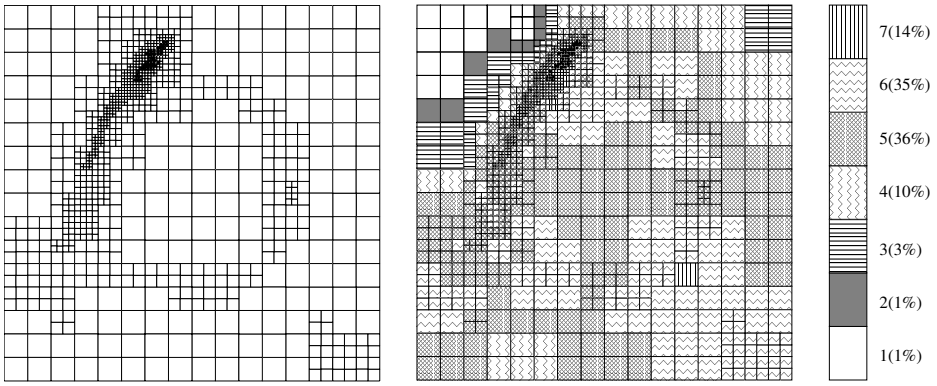


Figure 3. h- and hp-meshes after 11 refinements, with 1313 nodes, 1132 elements and 47406 degrees of freedom: here, $|J(u) - J(u_{\mathrm{DG}})| = 6.531 \times 10^{-10}$; from [25].

domain upstream of the point of interest, as well as in the circular region where the underlying partial differential equation changes type. Once the h-mesh has adequately captured the structure of the primal and dual solutions, the hp-adaptive algorithm performs p-refinement elsewhere in the domain of dependence of the point of interest.

5. High-dimensional transport-diffusion problems

We conclude by giving some pointers to recent results on stabilised sparse finite element methods for high-dimensional partial differential equations (1) with nonnegative characteristic form (cf. [57]). Such high-dimensional equations arise from a number of important applications in physics, chemistry, biology and finance. The origins of sparse tensor-product constructions and hyperbolic cross spaces can be traced back to Babenko [3] and Smolyak [56]; we refer to the papers of Temlyakov [61], DeVore, Konyagin & Temlyakov [20] for the study of high-dimensional approximation problems, to the works of Wasilkowski & Woźniakowski [62] and Novak & Ritter [46] for high-dimensional integration problems and associated complexity questions, to the paper of Zenger [63] for an early contribution to the numerical solution of high-dimensional elliptic equations, to the articles by von Petersdorff & Schwab [50] and Hoang & Schwab [27] for the analysis of sparse-grid methods for high-dimensional elliptic multiscale problems and parabolic equations, respectively, and to the recent survey article of Bungartz & Griebel [17].

Suppose that $\Omega = (0,1)^d$, $\Gamma_{\rm N} = \emptyset$ and $g_{\rm D} = 0$ in (3), and that the operator $\mathcal{L}$ in (1) has constant coefficients. In the simplest case, the construction of the finite element space $\hat{V}_0^L \subset \mathcal{H}$ begins by taking the tensor product of d copies of a finite element space of univariate hierarchical continuous piecewise linear functions ($p = 1$) on a uniform mesh of size $h_L = 2^{-L}$, $L \geq 1$. The resulting tensor-product space V_0^L has dimension $\dim(V_0^L) = \mathcal{O}(h_L^{-d})$. Clearly, the use of this space would lead to exponential growth of computational complexity for fixed h_L, as d increases. Thus, the idea is to reduce the complexity of the computation for large d by sparsifying the space V_0^L; the resulting sparse finite element space is denoted $\hat{V}_0^L$ and has only $\dim(\hat{V}_0^L) = \mathcal{O}(h_L^{-1}|\log h_L|^{d-1})$ degrees of freedom. The relevant result from [57], stated in the theorem below, is that, with a careful choice of the streamline-diffusion stabilisation parameter δ_L and assuming that $u \in \mathcal{H}^2(\Omega) \cap \mathcal{H}$, where $\mathcal{H}^2(\Omega) = \{v : D^\alpha v \in {\rm L}_2(\Omega),\ |\alpha|_\infty \leq 2\}$ is the space of functions with $\rm L_2$-bounded mixed second derivatives, one can ensure that this reduction of computational complexity is achieved at essentially no loss in accuracy in the streamline-diffusion finite element method compared to the case when the full tensor-product space V_0^L is used instead of the sparse space $\hat{V}_0^L$.

Theorem 5.1. *Let $f \in {\rm L}_2(\Omega)$, $c > 0$ and $u \in \mathcal{H}^2(\Omega) \cap \mathcal{H}$. Then, the following bound holds for the error $u - u_{\rm SD}$ between the analytical solution u of* (8) *and its sparse finite element approximation $u_{\rm SD} \in \hat{V}_0^L$, with $L \geq 1$ and $h = h_L = 2^{-L}$:*

$$
\begin{aligned}
&|||u - u_{\rm SD}|||_{\rm SD}^2 \\
&\qquad \leq C(u)\left\{|a|h_L^2 + h_L^4|\log_2 h_L|^{2(d-1)}\max\left(\frac{|a|}{h_L^2},\ \frac{d|b|}{h_L|\log_2 h_L|^{d-1}},\ c\right)\right\}
\end{aligned}
$$

with the streamline-diffusion parameter δ_L defined by the formula

$$\delta_L := K_\delta \min\left(\frac{h_L^2}{|a|}, \frac{h_L|\log_2 h_L|^{d-1}}{d|b|}, \frac{1}{c}\right),$$

with $K_\delta \in \mathbb{R}_{>0}$ a constant, independent of h_L and d, and $C(u) = \text{Const.}\|u\|^2_{\mathcal{H}^2(\Omega)}$ where Const. *is a positive constant independent of the discretisation parameter h_L.*

We refer to [57] for further technical details, including the proof of this result.

6. Concluding remarks

We surveyed continuous stabilised and discontinuous Galerkin finite element methods for the numerical solution of second-order partial differential equations with nonnegative characteristic form. We stated *a priori* and residual-based *a posteriori* error bounds, and in the case of the discontinuous Galerkin method we showed how the *a posteriori* bound may be used to drive an *hp*-adaptive finite element algorithm. We also commented on the use of sparse stabilised finite element methods for high-dimensional transport-dominated diffusion equations: stochastic analysis and kinetic theory are particularly fertile sources of Fokker–Planck equations of this kind [41]. The numerical solution of high-dimensional partial differential equations has been an active area of research in recent years [17], though the bulk of the research has been confined to self-adjoint elliptic and parabolic equations. As we have briefly indicated, extensions of these results to the, vastly richer, class of partial differential equations with nonnegative characteristic form are feasible, and we expect that activities in this direction will continue to flourish.

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