# Ergodic control of diffusion processes

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**Abstract.** Results concerning existence and characterization of optimal controls for ergodic control of nondegenerate diffusion processes are described. Extensions to the general 'controlled martingale problem' are indicated, which cover in particular degenerate diffusions and some infinite dimensional problems. In conclusion, some related problems and open issues are discussed.

**Mathematics Subject Classification (2000).** Primary 93E20; Secondary 60H20.

**Keywords.** Controlled diffusions, ergodic control, stationary Markov control, controlled martingale problems, dynamic programming.

## 1. Introduction

Ergodic or 'long run average' control of Markov processes considers the minimization of a time-averaged cost over admissible controls. This stands apart from the usual 'integral' cost criteria such as finite horizon or infinite horizon discounted cost criteria because neither the dynamic programming principle nor the usual 'tightness' arguments for existence of optima common to these set-ups carry over easily to the ergodic problem. Thus entirely new proof techniques have to be employed. The situation gets more complicated for continuous time continuous state space processes, of which diffusion processes are a prime example, because of the additional technicalities involved. This article describes first the reasonably well-understood case of non-degenerate diffusions, and then the partly resolved case of the more general 'controlled martingale problem' which covers degenerate diffusions and partially observed diffusions, among others.

An extended account of this topic will appear in [2].

## 2. Ergodic control of non-degenerate diffusions

**2.1. Preliminaries.** The $d$-dimensional ($d \geq 1$) controlled diffusion process $X(\cdot) = [X_1(\cdot), \dots, X_d(\cdot)]^T$ is described by the stochastic differential equation

$$X(t) = X_0 + \int_0^t m(X(s), u(s))\, ds + \int_0^t \sigma(X(s))\, dW(s), \tag{1}$$

*The work is supported in part by a grant from the Department of Science and Technology, Government of India

Proceedings of the International Congress
of Mathematicians, Madrid, Spain, 2006

for $t \geq 0$. Here:

1. for a compact metric 'control space' $U$, $m(\cdot,\cdot) = [m_1(\cdot,\cdot),\ldots,m_d(\cdot,\cdot)]^T : \mathcal{R}^d \times U \to \mathcal{R}^d$ is continuous and Lipschitz in the first argument uniformly with respect to the second,
2. $\sigma(\cdot) = [[\sigma_{ij}(\cdot)]]_{1\leq i,j\leq d} : \mathcal{R}^d \to \mathcal{R}^{d\times d}$ is Lipschitz,
3. $X_0$ is an $\mathcal{R}^d$-valued random variable with a prescribed law $\pi_0$,
4. $W(\cdot) = [W_1(\cdot),\ldots,W_d(\cdot)]^T$ is a $d$-dimensional standard Brownian motion independent of $X_0$,
5. $u(\cdot) : \mathcal{R}^+ \to U$ is the 'control process' with measurable paths, satisfying the *non-anticipativity condition*: for $t > s \geq 0$, $W(t) - W(s)$ is independent of $\{X_0, W(y), u(y), y \leq s\}$. (In other words, $u(\cdot)$ does not anticipate the future increments of $W(\cdot)$.)

This class of $u(\cdot)$ is referred to as *admissible* controls. It is known that without loss of generality, one may take these to be adapted to the natural filtration of $X(\cdot)$, given by $\mathcal{F}_t$ = the completion of $\bigcap_{s>t} \sigma(X(y), y \leq s)$. We shall say that it is a *stationary Markov* control if in addition $u(t) = v(X(t)), t \geq 0$, for a measurable $v : \mathcal{R}^d \to U$. By a standard abuse of terminology, we identify this control with the map $v(\cdot)$. We shall say that (1) is *non-degenerate* if the least eigenvalue of $\sigma(\cdot)\sigma^T(\cdot)$ is uniformly bounded away from zero, *degenerate* otherwise. We use the 'weak solution' framework, i.e., only the law of the pair $(X(\cdot), u(\cdot))$ is prescribed and 'uniqueness' is interpreted as uniqueness in law. For this section, we assume non-degeneracy. This in particular implies existence of a unique *strong* solution for stationary Markov controls.

We shall also need the relaxation of the notion of control process $u(\cdot)$ above to that of a *relaxed* control process. That is, we assume that $U = \mathcal{P}(U_0)$, the space of probability measures on $U_0$ with Prohorov topology, where $U_0$ is compact metrizable (whence so is $U$) and $m_i(\cdot,\cdot)$, $1 \leq i \leq d$, are of the form

$$m_i(x,u) = \int \bar{m}_i(x,y)u(dy), \quad 1 \leq i \leq d,$$

for some $\bar{m}_i : \mathcal{R}^d \times U_0 \to \mathcal{R}$ that are continuous and Lipschitz in the first argument uniformly w.r.t. the second. We may write $u(t) = u(t,dy)$ to underscore the fact that it is a measure-valued process. Likewise for stationary Markov controls, write $v(\cdot) = v(\cdot,dy)$. Then the original notion of $U_0$-valued control $u_0(\cdot)$ (say) corresponds to $u(t,dy) = \delta_{u_0(t)}(dy)$, the Dirac measure at $u_0(t)$, for all $t$. We call such controls as *precise* controls. Precise stationary Markov controls may be defined accordingly.

The objective of ergodic control is to minimize

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T E[k(X(t),u(t))]\,dt \tag{2}$$

(the average version), or to a.s. minimize

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T k(X(t), u(t))\, dt \tag{3}$$

(the 'almost sure' version). Here $k : \mathcal{R}^d \times U \to \mathcal{R}$ is continuous. In view of our relaxed control framework, we take it to be of the form $k(x, u) = \int \bar{k}(x, y)u(dy)$ for a continuous $\bar{k} : \mathcal{R}^d \times U_0 \to \mathcal{R}$. This cost criterion is popular in applications where transients are fast, hence negligible, and one is choosing essentially from among the attainable 'steady states'. As mentioned above, we consider the non-degenerate case first. Most of the results presented in the remainder of this section have been established for bounded coefficients in the original sources, but the extension to the Lipschitz coefficients (implying linear growth) is not difficult and appears in [2]. One usually assumes (and we do) that $k$ is bounded from below.

**2.2. Existence results.** Let $v(\cdot)$ be a stationary Markov control such that the corresponding $X(\cdot)$ is positive recurrent and therefore has a unique stationary distribution $\eta^v \in \mathcal{P}(\mathcal{R}^d)$. Define the corresponding ergodic occupation measure as $\mu^v(dx, dy) = \eta^v(dx)v(x, dy)$. Costs (2), (3) will then equal ('a.s.' in the latter case) $\int \bar{k}d\mu^v$. A key result is:

**Theorem 2.1** ([18]). *The set $\mathcal{G} = \{\mu^v : v(\cdot)$ stationary Markov$\}$ is closed convex in total variation norm topology, with its extreme points corresponding to precise stationary Markov controls.*

We can say much more: define the empirical measures $\{\nu_t\}$ by:

$$\int f d\nu_t = \frac{1}{t}\int_0^t \int f(X(s), y)u(s, dy)\, ds, \quad f \in C_b(\mathcal{R}^d \times U_0),\ t > 0.$$

Let $\bar{\mathcal{R}} = \mathcal{R}^d \cup \{\infty\} =$ the one point compactification of $\mathcal{R}^d$ and view $\nu_t$ as a random variable in $\mathcal{P}(\bar{\mathcal{R}} \times U_0)$ that assigns zero mass to $\{\infty\} \times U_0$.

**Theorem 2.2** ([16]). *As $t \to \infty$, almost surely*

$$\nu_t \to \{\nu : \nu(A) = a\nu'(A \cap (\{\infty\} \times U_0)) + (1-a)\nu''(A \cap (\mathcal{R}^d \times U_0)) \textit{ for all } A \textit{ Borel in } \bar{\mathcal{R}} \times U_0,\ \textit{with } a \in [0, 1],\ \nu' \in \mathcal{P}(\{\infty\} \times U_0), \nu'' \in \mathcal{G}\}.$$

There are two important special cases for which Theorem 2.1 allows us to reduce the control problem to the infinite dimensional linear programming problem of minimizing $\int \bar{k}d\mu$ over $\mathcal{G}$ and thereby deduce the existence of an optimal precise stationary Markov control for the 'a.s.' version of the ergodic control problem [16]:

1. under a suitable 'stability condition' (such as a convenient 'stochastic Liapunov condition') that ensures compactness of $\mathcal{G}$ and a.s. tightness of $\{\nu_t\}$, or,

2. under a condition that penalizes escape of probability mass to infinity, such as the 'near-monotonicity condition':

$$\liminf_{||x||\to\infty} \min_u k(x, u) > \beta,$$

where $\beta =$ the optimal cost.

The latter condition is often satisfied in practice. The 'average' version of the ergodic cost can be handled similarly, using the average empirical measures $\{\bar{\nu}_t\}$ defined via

$$\int f\, d\bar{\nu}_t = \frac{1}{t}\int_0^t E\bigg[\int f(X(s), y)u(s, dy)\bigg]\, ds, \quad f \in C_b(\mathcal{R}^d \times U_0),\ t > 0,$$

in place of $\{\nu_t\}$.

**2.3. Dynamic programming.** The standard approach to dynamic programming for ergodic control, inherited from earlier developments in the discrete time and state problems, is to treat it as a limiting case of the infinite horizon discounted cost problem as the discount vanishes. Hence we begin with the infinite horizon discounted cost

$$E\bigg[\int_0^\infty e^{-\alpha t} k(X(t), u(t))\, dt\bigg],$$

where $\alpha > 0$ is the discount factor. Define

$$Lf(x, u) = \langle \nabla f(x), m(x, u)\rangle + \frac{1}{2}\mathrm{tr}\big(\sigma(x)\sigma^T(x)\nabla^2 f(x)\big)$$

for $f \in C^2(\mathcal{R}^d)$. We may write $L_u f(x)$ for $Lf(u, x)$, treating $u$ as a parameter. The Hamilton–Jacobi–Bellman (HJB) equation for the 'value function'

$$V^\alpha(x) = \inf E\bigg[\int_0^\infty e^{-\alpha t} k(X(t), u(t))\, dt | X(0) = x\bigg]$$

(where the infimum is over all admissible controls) can be arrived at by standard dynamic programming heuristic and is

$$\min_u (k(x, u) - \alpha V^\alpha(x) + LV^\alpha(x, u)) = 0$$

on the whole space. For $k$ bounded from below, $V^\alpha$ is its least solution in $C^2(\mathcal{R}^d)$. Define $\overline{V}^\alpha = V^\alpha - V^\alpha(0)$. Then $\overline{V}^\alpha$ satisfies

$$\min_u (k(x, u) - \alpha\overline{V}^\alpha(x) - \alpha V^\alpha(0) + L\overline{V}^\alpha(x, u)) = 0. \tag{4}$$

Under suitable technical conditions (such as near-monotonicity or stability conditions mentioned above) one can show that as $\alpha \downarrow 0$, $\overline{V}^\alpha(\cdot)$ and $\alpha V^\alpha(0)$ converge along a subsequence to some $V, \beta$ in an appropriate Sobolev space and $\mathcal{R}$, respectively. Letting $\alpha \downarrow 0$ along this subsequence in (4), these are seen to satisfy

$$\min_u (k(x,u) - \beta + LV(x,u)) = 0.$$

This is the HJB equation of ergodic control. One can show uniqueness of $\beta$ as being the optimal ergodic cost and of $V$ up to an additive scalar in an appropriate function class depending on the set of assumptions one is working with. A verification theorem holds, i.e., the optimal stationary Markov control $v(\cdot)$ is characterized by the condition

$$v(x) \in \text{Argmin}\,(k(x,\cdot) + \langle \nabla V(x), m(x,\cdot)\rangle)\,, \text{ a.e.}$$

See [6], [17]. Note that the minimum will be attained in particular at a precise stationary Markov control, establishing the existence of an optimal precise stationary Markov control.

One also has the following stochastic representations for the ergodic value function $V$ (modulo an additive constant):

$$V(x) = \lim_{r \downarrow 0} \left( \inf E\left[ \int_0^{\tau_r} (k(X(s), u(s)) - \beta)\, ds | X(0) = x \right]\right),$$

where $\tau_r = \min\{t > 0 : ||X(t)|| = r\}$ for $r > 0$ [17] and the infimum is over all admissible controls. Alternatively,

$$V(x) = \inf \left( \inf_\tau E\left[ \int_0^{\tau} (k(X(s), u(s)) - \beta)\, ds | X(0) = x \right]\right),$$

where the inner infimum is over all bounded stopping times w.r.t. the natural filtration $\{\mathcal{F}_t\}$ of $X(\cdot)$, and the outer infimum is over all $\{\mathcal{F}_t\}$-adapted controls [21].

## 3. Controlled martingale problems

**3.1. Preliminaries.** Such explicit results are not as forthcoming in the more general scenario we discuss next. We shall denote by $E$ the Polish space that will serve as the state space of the controlled Markov process $X(\cdot)$, and by $U_0$ the compact metric 'control' space. $\mathcal{U}$ will denote the space of measurable maps $[0,\infty) \to U = \mathcal{P}(U_0)$ with the coarsest topology that renders continuous each of the maps

$$\mu(\cdot) = \mu(\cdot, du) \in \mathcal{U} \mapsto \int_0^T g(t) \int_{U_0} h(u)\mu(t, du)\, dt,$$

for all $T > 0$, $g \in L_2[0,T]$, $h \in C_b(U_0)$. This is compact metrizable (see, e.g., [9]). The control process $u(\cdot)$ can then be viewed as a $\mathcal{U}$-valued random variable.

For $\{f_k\}$, $f \in B(E) \stackrel{\text{def}}{=}$ the space of bounded measurable maps $E \to \mathcal{R}$, say that $f_k \xrightarrow{\text{bp}} f$ (where 'bp' stands for '*bounded pointwise*') if $\sup_{x,k} |f_k(x)| < \infty$ and

$f_k(x) \to f(x)$ for all $x$. $Q \subset B(E)$ is *bp-closed* if $f_k \in Q$ for all $k$ and $f_k \xrightarrow{\text{bp}} f$ together imply $f \in Q$. For $Q \subset B(E)$, define bp-closure$(Q)$ = the smallest bp-closed subset of $B(E)$ containing $Q$.

Let $A$ be an operator with domain $\mathcal{D}(A) \subset C_b(E)$ and range $\mathcal{R}(A) \subset C_b(E \times U_0)$. Let $\nu \in \mathcal{P}(E)$.

**Definition 3.1.** *An $E \times U$-valued process $(X(\cdot), \pi(\cdot) = \pi(\cdot, du))$ defined on a probability space $(\Omega, \mathcal{F}, P)$ is said to be a solution to the relaxed controlled martingale problem for $(A, \nu)$ with respect to a filtration $\{\mathcal{F}_t, t \geq 0\}$ if:*

- $(X(\cdot), \pi(\cdot))$ *is $\{\mathcal{F}_t\}$-progressive;*
- $\mathcal{L}(X(0)) = \nu$*;*
- *for $f \in \mathcal{D}(A)$,*

$$f(X(t)) - \int_0^t \int_{U_0} Af(X(s), u)\pi(s, du)\, ds, \quad t \geq 0, \tag{5}$$

*is an $\{\mathcal{F}_t\}$-martingale.*

We omit explicit mention of $\{\mathcal{F}_t\}$ or $\nu$ when they are apparent from the context. The operator $A$ is assumed to satisfy the following conditions:

1. (C1) There exists a countable subset $\{g_k\} \subset \mathcal{D}(A)$ such that

$$\{(g, Ag) : g \in \mathcal{D}(A)\} \subset \text{bp-closure}(\{(g_k, Ag_k) : k \geq 1\}).$$

2. (C2) $\mathcal{D}(A)$ is an algebra that separates points in $E$ and contains constant functions. Also, $A\mathbf{1} = 0$, where $\mathbf{1}$ is the constant function identically equal to 1.
3. (C3) For each $u \in U_0$, let $A^u f(\cdot) = Af(\cdot, u)$. Then there exists an r.c.l.l. solution to the martingale problem for $(A^u, \delta_x)$ for all $u \in U_0$, $x \in E$.

For example, the following can be shown to fit this framework:

1. $X(\cdot)$ as in (1) with or without the non-degeneracy condition.
2. An important instance of the above is the 'separated control problem' for control of diffusions with partial observations, which we describe in some detail next. Append to (1) the 'observation equation'

$$Y(t) = \int_0^t h(X(s))\, ds + W'(t),$$

where $h : \mathcal{R}^d \to \mathcal{R}^s$ $(s \geq 1)$ is a Lipschitz observation map and $W'(\cdot)$ is an $s$-dimensional standard Brownian motion independent of $(X_0, W(\cdot))$, representing the (integrated) observation noise. The control $u(\cdot)$ is ideally required to be adapted to the natural filtration of $Y(\cdot)$, but a standard relaxation allows for somewhat more general 'wide sense admissible' controls. These require merely that under a locally (in time) absolutely continuous change of measure that retains (1) but renders $Y(\cdot)$ itself an $s$-dimensional standard Brownian motion independent of $(X_0, W(\cdot))$, the future increments $Y(t+\cdot) - Y(t)$ should be independent of $\{X_0, W(\cdot), u(s), Y(s), s \leq t\}$ for all $t > 0$. The correct state variable for this problem (to be precise, one choice thereof) turns out to be the $\mathcal{P}(\mathcal{R}^d)$-valued process $\{\mu_t\}$ of regular conditional laws of $X(t)$ given $\{Y(s), u(s), s \leq t\}$ for $t \geq 0$. This evolves according to the equations of nonlinear filtering:

$$\mu_t(f) = \mu_0(f) + \int_0^t \mu_s(L_{u(s)} f)\, ds + \int_0^t \langle \mu_s(fh) - \mu_s(f)\mu_s(h), d\hat{Y}(s)\rangle \tag{6}$$

for $f \in C_b^2(\mathcal{R}^d)$, where we follow the notation $\nu(f) = \int f d\nu$. The products in the integrand of the stochastic integral in (6) are componentwise, and the process $\hat{Y}(t) = Y(t) - \int_0^t \mu_s(h) ds$, $t \geq 0$, is the so called 'innovations process' which is an $s$-dimensional standard Brownian motion that generates the same natural filtration as $Y(\cdot)$ [1]. The well-posedness of (6) can be established under additional regularity conditions on $h$ [24]. In terms of $\{\mu_t\}$, the ergodic cost can be rewritten as

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t E[\mu_s(\bar{k}(\cdot, u(s)))]\, ds.$$

The $\mathcal{P}(\mathcal{R}^d)$-valued controlled Markov process $\{\mu_t\}$ with this cost functional can be shown to fit the above framework. This is called the 'separated control problem' because it separates the issues of estimation and control.

3. Certain Hilbert-space valued controlled stochastic evolution equations can also be shown to fit the above framework [7].

**3.2. The control problem.** Let $k : E \times U_0 \to [0, \infty]$ be a continuous *running cost* function. The ergodic control problem is to minimize the *ergodic cost*

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t E\left[\int_{U_0} k(X(s), u)\pi(s, du))\right] ds. \tag{7}$$

We assume that the set of laws of $(X(\cdot), \pi(\cdot))$ for which this is finite is nonempty.

For a stationary $(X(\cdot), \pi(\cdot))$, define the associated *ergodic occupation measure* $\varphi \in \mathcal{P}(E \times U_0)$ by:

$$\int f(x, u)\varphi(dxdu) = E\left[\int_{U_0} f(X(t), u)\pi(t, du)\right].$$

Note that (7) then becomes $\int k d\varphi$. Let $\mathcal{G}$ denote the set of all ergodic occupation measures. From [7], we then have (see [27], [33], [34] for related results):

**Theorem 3.2.** *$\mathcal{G}$ is closed convex and is characterized as*

$$\mathcal{G} = \left\{ \mu \in \mathcal{P}(E \times U_0) : \int Af \, d\mu = 0 \text{ for all } f \in \mathcal{D}(A) \right\}.$$

*In particular, for each $\mu \in \mathcal{G}$, there exists a stationary pair $(X(\cdot), \pi(\cdot))$ whose marginal at each time is $\mu$. Furthermore, $\pi(\cdot)$ may be taken to be stationary Markov.*

This can be made a starting point for existence results in specific cases. For example, for degenerate diffusions and the separated control problem for partially observed diffusions, somewhat stronger variants of the 'stability' and 'near-monotonicity' conditions described earlier suffice for the existence of an optimal stationary pair $(X(\cdot), \pi(\cdot))$. By considering the ergodic decomposition thereof, 'stationary' here may be improved to 'ergodic' [7]. Also, in view of the above theorem, the control therein may be taken to be stationary Markov.

This, however, does not imply that the process $X(\cdot)$ itself is time-homogeneous Markov, or even Markov. To establish the existence of an optimal Markov solution, we assume the following:

*For a fixed initial law $\nu$ of $X_0$, the attainable laws of $(X(\cdot), \pi(\cdot))$ form a tight set*

$$\bar{\mathcal{M}}(\nu) \subset \mathcal{P}(D([0, \infty); E) \times \mathcal{U}).$$

Simple sufficient conditions for this can be given in specific cases mentioned above. An immediate consequence of this is that $\bar{\mathcal{M}}(\nu)$ is in fact a compact convex set. Consider the equivalence relation on $\bar{\mathcal{M}}(\nu)$ that equates two laws when the corresponding one dimensional marginals agree for a.e. $t$. The set of equivalence classes, called the 'marginal classes', then forms a convex compact set in the quotient topology.

**Theorem 3.3.** *Every representative of an extremal marginal class corresponds to a Markov process.*

This is proved for degenerate diffusions in [10] and for the separated control problem in [20], but the same arguments carry over to the general case. This can be combined with the above to deduce the existence of an optimal pair $(X(\cdot), \pi(\cdot))$ such that $\pi(\cdot)$ is stationary Markov and $X(\cdot)$ Markov, though not necessarily time-homogeneous Markov [7]. Also, $(X(\cdot), \pi(\cdot))$ need not be stationary. Our experience with the non-degenerate case, however, suggests the existence of a stationary ergodic time-homogeneous Markov solution that is optimal. Under additional technical conditions, such a result has been proved in [8] by stretching the 'vanishing discount' argument, but there is scope for improvement.

As for dynamic programming, scattered results are available in specific cases. The degenerate problem has been approached in the viscosity solution framework [3], [4], [5]. For the separated control problem under partial observations, a martingale dynamic programming principle has been derived [12], [13]. Dualizing the linear programme above yields the following dual linear programme that can be interpreted as 'dynamic programming inequalities' [7]:

*Maximize* $z \in \mathcal{R}$ *subject to* $Lf(x,u) + k(x,u) \geq z$, *for all* $x \in E$, $u \in U_0$, $f \in \mathcal{D}(L)$.

## 4. Some related problems and open issues

1. 'Ergodic control with constraints' seeks to minimize one ergodic cost functional while imposing bounds on one or more additional ergodic cost functionals. In the linear programming formulation alluded to above, this amounts to a few additional constraints. Existence of optimal precise stationary Markov controls has been proved in the non-degenerate case under suitable stability or near-monotonicity hypotheses [11], [18]. A Lagrange multiplier formulation can be used to aggregate the costs into a single cost.

2. We did not include control in the diffusion matrix $\sigma(\cdot)$. The reason for this is that, for stationary Markov controls $u(\cdot) = v(X(\cdot))$, one is in general obliged to consider at best measurable $v(\cdot)$. For a merely measurable diffusion matrix, even in the non-degenerate case only the existence of a weak solution is available, the uniqueness may not hold [26] (except in one and two dimensions – see [35], pp. 192–194). It may, however, be possible to work with 'the set of all weak solutions' in place of '*the*' solution, but this is not very appealing unless one has a good selection criterion that prescribes a unique choice from among the many.

3. Singularly perturbed ergodic control concerns ergodic control of diffusions wherein some components move on a much faster time scale, characterized by a perturbation parameter $\epsilon > 0$. One can show that as $\epsilon \downarrow 0$, the slower components satisfy an 'averaged' dynamics wherein the coefficients in their dynamics are averaged over the stationary distribution of the fast components when the latter is derived by 'freezing' the slower components to constant values. The ergodic control problem for this limiting case is then a valid approximation for the original problem for small $\epsilon > 0$. See [15] for a precise statement and proofs.

4. We have not considered several related problems with a similar flavor, such as ergodic control of reflected [14] or switching diffusions [23], [30], ergodic impulse control [32], singular ergodic control [31], and stochastic differential games with ergodic payoffs [19]. The latter in particular are also of interest in risk-sensitive control problems on infinite time horizon, which effectively get converted to two person zero sum stochastic differential games with ergodic payoffs after the celebrated 'log-transform' of the value function [22].

5. We have also not addressed the computational issues here. Two major strands therein are Markov chain approximations [28] and approximations of the infinite dimensional linear programmes [25].

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