

Controllability of evolution equations of fluid dynamics

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Abstract. In this paper we will discuss recent developments in controllability of evolution equations of fluid mechanics. The control is assumed to be distributed either on a part of the boundary or locally distributed in some subdomain. We will present some ideas of proof of main theorems. Special attention will be paid to the technique based on Carleman estimates.

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1. Introduction

This paper is concerned with the problem of exact controllability of partial differential equations with control concentrated either on the part of the boundary or locally distributed inside of the boundary in some subdomain. The typical statement of general controllability problem, which we are going to discuss in this paper, can be formulated as follows: let a function $y(t, x)$, which describes the state of a system, satisfy a semilinear partial differential equation

$$\partial_t y + A(x, D)y + F(x, y, \nabla y) = \chi_\omega u \quad \text{in } (0, T) \times \Omega, \quad (1.1)$$

$$B(x, D)y = 0 \quad \text{on } (0, T) \times \partial\Omega, \quad y(0, \cdot) = y_0, \quad (1.2)$$

where $A(x, D)$ is a linear operator, F is a nonlinear term, $B(x, D)$ is a boundary operator, χ_ω is the characteristic function of the domain $\omega \subset \Omega$ where the control function $u(t, x)$ is supported.

The initial conditions, the function $y_0(x)$ and another function $y_1(x)$ called target function, are given. Let us choose some time moment T . Then the exact controllability problem may be formulated as follows: Find the control u and the state function y such that

$$y(T, \cdot) = y_1. \quad (1.3)$$

The solvability properties of the controllability problem (1.1)–(1.3) are completely different from the properties of the initial boundary value problem (1.1)–(1.2) with fixed u . For initial value problems with a reasonable choice of boundary conditions and for smooth initial conditions we usually expect the uniqueness of the solution. Moreover, if a priori estimates are obtained, this solution typically may be extended

globally in time. On the other hand, for most boundary/locally distributed controllability problems of equations of mathematical physics, solutions are not unique and a priori estimates typically are absent. In case of control of linear equations these difficulties do not produce a huge problem, since the controllability problem typically can be reduced to an *observability* problem which can be formulated as follows. Suppose two Banach spaces X and Y are given. For the solution of the adjoint linear equation

$$-\partial_t z + A^*(x, D)z = 0 \quad \text{in } (0, T) \times \Omega, \quad (1.4)$$

$$B^*(x, D)z = 0 \quad \text{on } (0, T) \times \partial\Omega \quad (1.5)$$

one needs to obtain the a priori estimate

$$\|z\|_X \leq C \|\chi_\omega z\|_Y. \quad (1.6)$$

The initial conditions at $t = T$ for problem (1.4), (1.5) are not assumed to be known. This creates the main difficulty in proving estimate (1.6). There are several methods to deal with such observability problems:

1. The method based on the theorem on propagation of singularities (see Bardos–Lebeau–Rauch [2]);
2. Multipliers method (see [31], [28], [27], [29], [39]);
3. Carleman estimates (see [21], [22], [25], [26], [37], [38]).

The first two methods are effective for the wave and Schrödinger equations. As for the equations of parabolic type and the generalized Stokes system, the Carleman type estimates with the singular weight functions appears to be more effective method compared to methods 1 and 2.

2. Controllability of parabolic equations and the Burgers equation

In a bounded domain $\Omega \in \mathbb{R}^N$ with $\partial\Omega \in C^2$ we consider the semilinear parabolic equation

$$G(y) = \partial_t y - \Delta y + f(t, x, y) = \chi_\omega u + g \quad (2.7)$$

with given initial condition and zero Dirichlet boundary conditions

$$y|_{(0,T) \times \partial\Omega} = 0, \quad y(0, \cdot) = y_0. \quad (2.8)$$

Here ω is an arbitrary but fixed subdomain, χ_ω is the characteristic function of the domain ω and $u(t, x)$ is the control locally distributed in ω . Suppose that the target function $y_1(x)$ is given and some moment of time T is fixed. We are looking for control u such that

$$y(T, \cdot) = y_1. \quad (2.9)$$

Since the solution $y(t, x)$ of the heat equation with zero right-hand side is analytic as a function of x for any positive t , we cannot solve in general problem (2.7)–(2.9) for an arbitrary smooth target function y_1 .

Let us assume that

$$f \in C^1([0, T] \times \bar{\Omega} \times \mathbb{R}^1), \quad f(t, x, 0) = 0 \quad \text{for all } (t, x) \in (0, T) \times \Omega, \quad (2.10)$$

and that the function $f(t, x, y)$ satisfies the Lipschitz condition

$$|f(t, x, \zeta_1) - f(t, x, \zeta_2)| \leq K|\zeta_1 - \zeta_2| \quad \text{for all } (t, x) \in (0, T) \times \Omega, \quad \zeta_1, \zeta_2 \in \mathbb{R}^1, \quad (2.11)$$

where the constant K is independent of t, x, ζ .

We have

Theorem 2.1 ([22]). *Let $y_1 \equiv 0$ and conditions (2.10), (2.11) hold true. Suppose that there exists $\delta > 0$ such that $e^{\frac{1}{(T-t)^{1+\delta}}} g \in L^2((0, T) \times \Omega)$. Then for any $y_0 \in \dot{W}_2^1(\Omega)$, there exists a solution $(y, u) \in W_2^{1,2}((0, T) \times \Omega) \times L^2((0, T) \times \omega)$ to problem (2.7)–(2.9).*

Here $W^{1,2}((0, T) \times \Omega) = \{y(t, x) \mid \partial_t y, \partial_x^\beta y \in L^2((0, T) \times \Omega) \text{ for all } |\beta| \leq 2\}$.

Thanks to assumption (2.11) by standard methods of functional analysis the proof of Theorem 2.1 may be reduced to the question of solvability of the controllability problem for the linear parabolic equation

$$\begin{aligned} \partial_t v - \Delta v + c(t, x)v &= \chi_\omega \tilde{u} + \tilde{g} \quad \text{in } (0, T) \times \Omega, \\ v|_{(0,T) \times \partial\Omega} &= 0, \quad v(0, \cdot) = v_0, \quad v(T, \cdot) = 0, \end{aligned} \quad (2.12)$$

where $c \in L^\infty((0, T) \times \Omega)$. The solvability of problem (2.12) is equivalent to obtaining the observability estimate for the adjoint parabolic equation:

$$-\partial_t z - \Delta z + c(t, x)z = q \quad \text{in } [0, T] \times \Omega, \quad (2.13)$$

$$z|_{(0,T) \times \partial\Omega} = 0. \quad (2.14)$$

The observability estimate for (2.13)–(2.14) can be proved using the technique of Carleman estimates. First we need to introduce some weight functions. Let $\psi \in C^2(\bar{\Omega})$ be such that

$$\psi(x) > 0 \quad \text{for all } x \in \Omega, \quad \psi|_{\partial\Omega} = 0, \quad |\nabla\psi(x)| > 0 \quad \text{for all } x \in \Omega \setminus \omega_0, \quad (2.15)$$

where $\omega_0 \subset\subset \omega$ is some open set. Using the function ψ we construct three more functions: $\varphi(t, x) = e^{\lambda\psi(x)}/(t(T-t))$, $\alpha(t, x) = (e^{\lambda\psi} - e^{2\lambda\|\psi\|_{C(\bar{\Omega})}})/(t(T-t))$, and $\eta(t, x) = (e^{\lambda\psi} - e^{2\lambda\|\psi\|_{C(\bar{\Omega})}})/(\ell(t)(T-t))$ where $\ell \in C^\infty[0, T]$, $\ell(t) > 0$ for any $t \in [0, T]$ and $\ell(t) = t$ for $t \in [\frac{3T}{4}, T]$.

The following holds:

Lemma 2.1 ([22]). *There exists a number $\hat{\lambda} > 0$ such that for an arbitrary $\lambda \geq \hat{\lambda}$ there exists $s_0(\lambda)$ such that for each $s \geq s_0(\lambda)$ solutions to problem (2.13)–(2.14)*

satisfy the following inequality:

$$\int_{(0,T) \times \Omega} \left(\frac{1}{s\varphi} \left(\left| \frac{\partial z}{\partial t} \right|^2 + |\Delta z|^2 \right) + s\varphi |\nabla z|^2 + s^3 \varphi^3 z^2 \right) e^{2s\alpha} dx dt \quad (2.16)$$

$$\leq C \left(\int_{(0,T) \times \Omega} |q|^2 e^{2s\alpha} dx dt + \int_{[0,T] \times \omega} s^3 \varphi^3 z^2 e^{2s\alpha} dx dt \right),$$

where the constant C is independent of s .

This estimate, combined with the standard energy estimate for equation (2.13), implies that for any $v_0 \in \mathring{W}_2^1(\Omega)$ and $e^{-s_0\eta} \tilde{g} \in L^2((0, T) \times \Omega)$ there exists a solution to problem (2.12): a pair $(y, u) \in W^{1,2}((0, T) \times \Omega) \times L^2((0, T) \times \omega)$ such that $e^{-s_0\eta} \tilde{u} \in L^2((0, T) \times \Omega)$, $e^{-s_0\eta} v / (T - t)^{\frac{3}{2}} \in L^2((0, T) \times \Omega)$.

A different approach, still based on Carleman estimates, was proposed by G. Lebeau and L. Robbiano in [30] for linear parabolic equations with time independent coefficients. In [8], [11], [35] solutions for the controllability problem of the linear heat equation were constructed directly by solving a moment problem. In [36] the solution to the controllability problem for the heat equation was obtained from a solution of the corresponding problem for the wave equation. In [34] another method was proposed, essentially based on the solvability of the Cauchy problem for the one dimensional heat equation. Later this method was applied to the semilinear parabolic equation in [33]. The approximate controllability for equation (2.7) was proved in [9].

Next we consider the problem of exact controllability of equation (2.7) with boundary control. Let Γ_0 be an arbitrary subdomain of $\partial\Omega$. Suppose that the control u is distributed over Γ_0 :

$$G(y) = g, \quad y|_{(0,T) \times \Gamma_0} = u, \quad y|_{(0,T) \times \partial\Omega \setminus \Gamma_0} = 0, \quad y(0, \cdot) = y_0, \quad y(T, \cdot) = y_1. \quad (2.17)$$

We have

Theorem 2.2 ([22]). *Let $y_1 \equiv 0$ and conditions (2.10), (2.11) hold true. Suppose that there exists $\delta > 0$ such that $e^{\frac{1}{(T-t)^{1+\delta}}} g \in L^2((0, T) \times \Omega)$. Then for any $y_0 \in \mathring{W}_2^1(\Omega)$ there exists a solution $(y, u) \in W_2^{1,2}((0, T) \times \Omega) \times L^2(0, T; H^{\frac{1}{2}}(\Gamma_0))$ to problem (2.17).*

Theorem 2.2 will easily follow from Theorem 2.1 if we enlarge the domain Ω up to $\tilde{\Omega}$ in such a way that

$$\omega \subset \tilde{\Omega}, \quad \omega = \tilde{\Omega} \setminus \Omega, \quad \partial\omega \cup \partial\Omega \subset \Gamma_0.$$

Then we consider problem (2.7)–(2.9) in $\tilde{\Omega}$ with the control locally distributed in ω . Since the existence of the solution y is guaranteed by Theorem 2.1 we consider the restriction of y on Ω and put $u = y|_{\Gamma_0}$.

Next we consider the situation when the target function is not zero. In order to solve the controllability problem we need some conditions on the functions y_1 and g .

Condition 2.1. There exists a constant $\tau > 0$ and a function $\tilde{u} \in L^2((0, T) \times \omega)$ such that the boundary value problem

$$G(\tilde{y}) = \chi_\omega \tilde{u} + g \text{ in } [T - \tau, T] \times \Omega, \quad \tilde{y}|_{[T-\tau, T] \times \partial\Omega} = 0, \quad \tilde{y}(T, \cdot) = y_1$$

has a solution $\tilde{y} \in W^{1,2}((0, T) \times \Omega)$.

We have

Theorem 2.3 ([22]). *Let $y_0 \in \mathring{W}_2^1(\Omega)$ and $g \in L^2((0, T) \times \Omega)$. Suppose that (2.10), (2.11) hold true. Let the functions y_1 and g satisfy Condition 2.1. Then there exists a solution $(y, u) \in W^{1,2}((0, T) \times \Omega) \times L^2((0, T) \times \omega)$ of problem (2.7)–(2.9).*

Theorem 2.3 provides necessary and sufficient conditions for solvability of problem (2.7)–(2.9).

A similar result holds true for the situation when the control is locally distributed over the boundary.

Condition 2.2. There exists a constant $\tau > 0$ and a function $\tilde{u} \in L^2((0, T) \times \omega)$ such that the boundary value problem

$$G(\tilde{y}) = g \text{ in } [T - \tau, T] \times \Omega, \\ \tilde{y}|_{[T-\tau, T] \times \Gamma_0} = \tilde{u}, \quad \tilde{y}|_{[T-\tau, T] \times \partial\Omega \setminus \Gamma_0} = 0, \quad \tilde{y}(T, \cdot) = y_1$$

has a solution $\tilde{y} \in W^{1,2}((0, T) \times \Omega)$.

The following holds:

Theorem 2.4 ([22]). *Let $y_0 \in \mathring{W}_2^1(\Omega)$ and $g \in L^2((0, T) \times \Omega)$. Suppose that (2.10), (2.11) hold true. Let the functions y_1 and g satisfy Condition 2.2. Then there exists a solution $(y, u) \in W^{1,2}((0, T) \times \Omega) \times L^2(0, T; H^{\frac{1}{2}}(\partial\Omega))$ of problem (2.17).*

In case when the nonlinear term of the parabolic equation is superlinear the situation is different. For example, there exists $y_0 \in C^\infty(\bar{\Omega})$ and a time moment \hat{T} which depends on Ω only, such that any solution for the initial value problem

$$\partial_t y - \Delta y + y^2 = 0 \text{ in } \Omega, \quad y(0, \cdot) = y_0, \quad y|_{(0, T) \times \partial\Omega} = u$$

will blow up at some time $\tau(u) < \hat{T}$. Hence we even cannot prevent a blowup by the boundary control. The similar question for the nonlinearity $f(t, x, y) = -y^3$ is open. If the nonlinear term has the form $f(t, x, y) = y^3$ for any $y_0 \in W_2^1(\Omega) \cap L^6(\Omega)$ and sufficiently regular u (which satisfies the compatibility condition) a solution to the initial value problem

$$\partial_t y - \Delta y + y^3 = 0 \text{ in } \Omega, \quad y(0, \cdot) = y_0, \quad y|_{(0, T) \times \partial\Omega} = u$$

exists and satisfies the a priori estimate

$$\frac{d}{dt} \int_\Omega \rho^7(x) y^2(t, x) dx + \frac{1}{8} \int_\Omega \rho^7(x) y^4(t, x) dx \leq C,$$

where $\rho \in C^2(\bar{\Omega})$ is an arbitrary function such that $\rho(x) > 0$ for each $x \in \Omega$, $\rho|_{\partial\Omega} = 0$, $|\nabla\rho|_{\partial\Omega} \neq 0$ and the constant C depends on ρ only. This estimate immediately implies that for some open set of target functions $y_1(x)$ in $L^2(\Omega)$ there is no solution to problem (2.17).

Let us consider the Burgers equation

$$\partial_t y - \partial_x^2 y + \partial_x y^2 = \chi_\omega u(t, x), \quad (t, x) \in [0, T] \times [0, L], \quad (2.18)$$

with zero Dirichlet boundary conditions and the initial condition

$$y(t, 0) = y(t, L) = 0, \quad y(0, \cdot) = y_0. \quad (2.19)$$

Here $\omega \subset [0, L]$ is an arbitrary but fixed open set. We are looking for a control u such that

$$y(T, \cdot) = y_1 \quad (2.20)$$

The following holds:

Theorem 2.5 ([16]). *Let $y_1 \in \mathring{W}_2^1(0, L)$ be a steady-state solution to the Burgers equation and $y_0 \in \mathring{W}_2^1(0, L)$. Then there exists a time moment $T(y_1)$ such that the controllability problem (2.18)–(2.20) has a solution $(y, u) \in W^{1,2}((0, T) \times [0, L]) \times L^2((0, T) \times [0, L])$.*

Suppose that ω satisfies the following condition:

$$\text{there exists } b > 0 \text{ such that } \omega \subset (b, L). \quad (2.21)$$

We have

Lemma 2.2 ([16]). *Let $y(t, x)$ be a solution to problem (2.18), (2.19). Denote $y_+(t, x) = \max(y(t, x), 0)$. Then for arbitrary $N > 5$ the following estimate holds true:*

$$\frac{d}{dt} \int_0^b (b-x)^N y_+^4(t, x) dx < \gamma(N) b^{N-5}. \quad (2.22)$$

Here $\gamma(N) > 0$ is a constant depending on N only.

The immediate consequence of (2.22) is the existence of an open set of target functions which is unreachable by means of the locally distributed control satisfying (2.21) or by means of the boundary control concentrated at $x = L$.

If condition (2.21) fails, we of course do not have the a priori estimate (2.22). In terms of the boundary control this situation corresponds to the case when the control is located at both endpoints of the segment $[0, L]$. By Hopf's transformation this problem might be reduced to the controllability problem of the one-dimensional heat equation with control located at both endpoints of the segment $[0, L]$ but with one additional constraint: control functions are nonnegative. Then from results of [1] it follows that for some initial condition y_0 the set of all reachable functions is not dense in $L^2(0, L)$. Later we will see that the controllability properties of the Burgers equation and the Navier–Stokes system are completely different.

3. Local controllability of the Navier–Stokes system

In [32] J.-L. Lions conjectured that the Navier–Stokes system with boundary or locally distributed control is globally approximately controllable. This paper inspired intensive research in the area. In this section we discuss the local controllability results for the Navier–Stokes system and the Boussinesq system.

Let us consider the Navier–Stokes system defined on the bounded domain $\Omega \subset \mathbb{R}^N$ ($N = 2, 3$) with boundary $\partial\Omega \in C^2$

$$\partial_t y(t, x) - \Delta y(t, x) + (y, \nabla) y + \nabla p = f + \chi_\omega u \quad \text{in } \Omega, \quad \operatorname{div} y = 0, \quad (3.23)$$

$$y|_{(0,T) \times \partial\Omega} = 0, \quad y(0, \cdot) = y_0, \quad (3.24)$$

where $y(t, x) = (y_1(t, x), \dots, y_N(t, x))$ is the velocity of fluid, p is the pressure. The density of external forces $f(t, x) = (f_1(t, x), \dots, f_N(t, x))$ and the initial velocity y_0 are given, $u(t, x)$ is a control distributed in some arbitrary but fixed subdomain ω of the domain Ω .

Let $(\hat{y}(t, x), \hat{p}(t, x))$ be a solution of the Navier–Stokes equations with the right-hand side f exactly the same as in (3.23):

$$\partial_t \hat{y} - \Delta \hat{y} + (\hat{y}, \nabla) \hat{y} + \nabla \hat{p} = f \quad \text{in } (0, T) \times \Omega, \quad \operatorname{div} \hat{y} = 0, \quad \hat{y}|_{(0,T) \times \partial\Omega} = 0 \quad (3.25)$$

close enough to the initial condition y_0 at the moment $t = 0$

$$\|y_0 - \hat{y}(0, \cdot)\|_V \leq \varepsilon, \quad (\text{the parameter } \varepsilon \text{ is sufficiently small}) \quad (3.26)$$

where $V = \{y(x) = (y_1, \dots, y_N) \in (W_2^1(\Omega))^N : \operatorname{div} y = 0 \text{ in } \Omega, y|_{\partial\Omega} = 0\}$.

We are looking for a control u such that, for a given $T > 0$, the following equality holds

$$y(T, \cdot) = \hat{y}(T, \cdot). \quad (3.27)$$

In order to formulate our results, we introduce the following functional spaces:

$$H = \{y(x) = (y_1, \dots, y_N) \in (L^2(\Omega))^N : \operatorname{div} y = 0, \text{ in } \Omega, (y, \vec{n})|_{\partial\Omega} = 0\},$$

$$V^{1,2}((0, T) \times \Omega) = \{y(t, x) \in (W_2^{1,2}((0, T) \times \Omega))^N : \operatorname{div} y = 0 \text{ in } \Omega, y|_{\partial\Omega} = 0\},$$

where $\vec{n} = \vec{n}(x) = (n_1(x), \dots, n_N(x))$ is the outward unit normal to $\partial\Omega$.

Suppose that the function \hat{y} has the following regularity properties:

$$\begin{aligned} \hat{y} &\in L^\infty((0, T) \times \Omega), \\ \partial_t \hat{y} &\in L^2(0, T; L^\sigma(\Omega)), \quad \sigma > 6/5 \text{ for } N = 3, \sigma > 1 \text{ for } N = 2. \end{aligned} \quad (3.28)$$

The following result in particular gives us a positive answer to the question of the possibility of stabilization of the flow near an unstable steady state solution by means of locally distributed control.

Theorem 3.1 ([12]). *Let $y_0 \in V$, $f \in L^2(0, T; H)$ and suppose that the pair (\hat{y}, \hat{p}) solves (3.25) and satisfies condition (3.28). Then for sufficiently small $\varepsilon > 0$ there exists a solution $(y, p, u) \in V^{1,2}((0, T) \times \Omega) \times L^2(0, T; W_2^1(\Omega)) \times (L^2((0, T) \times \omega))^N$ to problem (3.23), (3.24), (3.26), (3.27).*

This result first has been proved in [15] for the control distributed over the whole boundary $\partial\Omega$. In [21] the case of control distributed over an arbitrary small subdomain ω , but with some assumptions on the geometry of Ω was considered. Finally, in [23], these assumptions on Ω were removed under the regularity condition on the function \hat{y} which is stronger than (3.28).

Since the existence theorem 3.1 is local, in order to prove this existence result one first proves the solvability of the controllability problem for the Navier–Stokes equation linearized at trajectory \hat{y} :

$$\begin{cases} \partial_t \tilde{y} - \Delta \tilde{y} + (\hat{y}, \nabla) \tilde{y} + (y, \nabla) \hat{y} + \nabla \tilde{p} = f + \chi_\omega \tilde{u}, \operatorname{div} \tilde{y} = 0 & \text{in } (0, T) \times \Omega, \\ \tilde{y} = 0 & \text{on } (0, T) \times \partial\Omega, \\ \tilde{y}(0, \cdot) = y_0, \tilde{y}(T, \cdot) = 0 & \text{in } \Omega. \end{cases} \tag{3.29}$$

After the solvability of (3.29) is established in appropriate functional spaces the conclusion of the Theorem 3.1 follows from the standard implicit function theorem.

The typical way to solve (3.29) is to reduce it to the observability problem for the operator adjoint to the operator of the linearization of the Navier–Stokes system at trajectory \hat{y} . More precisely, let the function $z \in L^2(0, T; H)$ satisfy the equations

$$-\partial_t z - \Delta z - Dz\hat{y} = \nabla \pi + g \text{ in } (0, T) \times \Omega, \tag{3.30}$$

$$\operatorname{div} z = 0, \quad z|_{(0,T) \times \partial\Omega} = 0, \tag{3.31}$$

where the function $Dz = \nabla z + \nabla z^t$.

Denote $\alpha(t, x) = \frac{e^{\lambda\psi(x)+8\|\psi\|_{L^\infty(\Omega)}} - e^{10\lambda\|\psi\|_{L^\infty(\Omega)}}}{(t(T-t))^4}$, $\alpha^*(t) = \min_{x \in \Omega} \alpha(t, x)$, $\hat{\alpha}(t) = \max_{x \in \Omega} \alpha(t, x)$, $\hat{\varphi}(t, x) = \frac{e^{8\lambda\|\psi\|_{L^\infty(\Omega)}}}{(t(T-t))^4}$, $\varphi(t, x) = \frac{e^{8\lambda\|\psi\|_{L^\infty(\Omega)} + \psi(x)}}{(t(T-t))^4}$. The function ψ is introduced in (2.15). For the system (3.30)–(3.31), we have the following observability estimate:

Theorem 3.2 ([12]). *There exist three positive constants $\hat{s}, \hat{\lambda}, C$ depending on Ω and ω such that for every $z_0 \in H$, $g \in L^2((0, T) \times \Omega)$ the corresponding solution to (3.30), (3.31) verifies:*

$$\begin{aligned} & \int_{(0,T) \times \Omega} \left(\frac{1}{s\varphi} \left(\left| \frac{\partial z}{\partial t} \right|^2 + \sum_{i,j=1}^n \left| \frac{\partial^2 z}{\partial x_i \partial x_j} \right|^2 \right) + s\lambda^2 \varphi |\nabla z|^2 + s^3 \lambda^4 \varphi^3 |z|^2 \right) e^{2s\alpha} dx dt \\ & \leq C(1 + T^2) \left(s^{\frac{15}{2}} \lambda^{20} \int_{(0,T) \times \Omega} |g|^2 \hat{\varphi}^{\frac{15}{2}} e^{4s\hat{\alpha} - 2s\alpha^*} dx dt \right. \\ & \quad \left. + \int_{(0,T) \times \omega} s^{16} \lambda^{40} \hat{\varphi}^{16} |z|^2 e^{8s\hat{\alpha} - 6s\alpha^*} dx dt \right) \text{ for all } s \geq s_0, \end{aligned} \tag{3.32}$$

for all $\lambda \geq \hat{\lambda}(1 + \|\hat{y}\|_{L^\infty((0,T)\times\Omega)}^2 + \|\partial_t \hat{y}\|_{L^2(0,T;L^\sigma(\Omega))}^2 + e^{\hat{\lambda}T\|\hat{y}\|_{L^\infty((0,T)\times\Omega)}^2})$ and $s \geq \hat{s}(T^4 + T^8)$.

The strategy of the proof of (3.32) is as follows. First we apply the Carleman estimate (2.16) to equations (3.31). Next we need to eliminate the norm of the function $\nabla\pi$ on the right-hand side. In order to do that we observe that the pressure π for each $t \in [0, T]$ satisfies the Laplace equation

$$-\Delta\pi = \operatorname{div}(Dz\hat{y}) + \operatorname{div} g \quad \text{in } \Omega. \tag{3.33}$$

Since the velocity field z satisfies the zero Dirichlet boundary conditions, there are no explicit boundary conditions for the pressure π . Therefore to equation (3.33) we apply the Carleman estimates for elliptic equations obtained in [24] with weights which minimize the contribution of the boundary terms. Finally we eliminate the norms of the functions $\pi|_{\partial\Omega}$ and $\chi_\omega\pi$ using some a priori estimates for the initial value problems for the Stokes system and the heat equation.

In many controllability problems in addition to be locally distributed in a subdomain, the control u is required to satisfy some additional constraints. Below we discuss the situation when in problems (3.23), (3.24), (3.27) the control satisfies the following constraint: one of the components of the vector function $u(t, x)$ is identically equal zero on $(0, T) \times \Omega$. Suppose that ω satisfies the following condition:

$$\text{there exists } x^0 \in \partial\Omega, \tilde{\delta} > 0 \text{ such that } \bar{\omega} \cap \partial\Omega \supset B(x^0; \tilde{\delta}) \cap \partial\Omega. \tag{3.34}$$

($B(x^0; \tilde{\delta})$ is the ball centered at x^0 of radius $\tilde{\delta}$.)

Let $E = H$ for $N = 2$ and $E = H \cap L^4(\Omega)$ for $N = 3$. Assume that the initial condition y_0 is close to $\hat{y}(0, \cdot)$ in the norm of the space E :

$$\|y_0 - \hat{y}(0, \cdot)\|_E \leq \varepsilon. \tag{3.35}$$

We have

Theorem 3.3. *Assume that ω satisfies (3.34). Let $y_0 \in E, f \equiv 0$ and suppose that the pair (\hat{y}, \hat{p}) solves (3.25) and satisfies condition (3.28). Then for sufficiently small $\varepsilon > 0$ there exists a solution (y, p, u) to problem (3.23), (3.24), (3.35), (3.27) with control $u \in (L^2((0, T) \times \omega))^N$ having one component identically zero.*

In the case of locally distributed control with zero component u_k for the corresponding observability problem, associated with (3.30),(3.31) we do not have any information on the k -th component of the function z in $(0, T) \times \omega$. This means that the function z_k should not appear in the right-hand side of the inequality (3.32). This difficulty can be overcome if we recall that z is divergence free function and therefore its k -th component satisfies the equation $\partial_{x_k} z_k = \sum_{j=1, j \neq k}^N \partial_{x_j} z_j$. From this ordinary differential equation, thanks to zero Dirichlet boundary conditions and assumption (3.34), in some subdomain of ω we can estimate z_k by the remaining components of the function z and then apply (3.32).

Next we consider the similar controllability problem of the Boussinesq system.

$$\begin{cases} \partial_t y - \Delta y + (y, \nabla)y + \nabla p = \chi_\omega u + \theta e_N, & \operatorname{div} y = 0 & \text{in } (0, T) \times \Omega, \\ \partial_t \theta - \Delta \theta + (y, \nabla)\theta = \chi_\omega h & & \text{in } (0, T) \times \Omega, \\ y = 0, \theta = 0 & & \text{on } (0, T) \times \partial\Omega, \\ y(0, \cdot) = y_0, \theta(0, \cdot) = \theta_0 & & \text{in } \Omega. \end{cases} \quad (3.36)$$

In the domain $(0, T) \times \omega$ we control both the density of external forces u and the density of external heat sources h .

Let $(\hat{y}, \hat{p}, \hat{\theta})$ be a sufficiently regular solution to the Boussinesq system:

$$\partial_t \hat{y} - \Delta \hat{y} + (\hat{y}, \nabla)\hat{y} + \nabla \hat{p} = \hat{\theta} e_N, \quad \operatorname{div} \hat{y} = 0 \quad \text{in } (0, T) \times \Omega, \quad (3.37)$$

$$\partial_t \hat{\theta} - \Delta \hat{\theta} + (\hat{y}, \nabla)\hat{\theta} = 0 \quad \text{in } (0, T) \times \Omega, \quad (3.38)$$

$$\hat{y} = 0, \hat{\theta} = 0 \quad \text{on } (0, T) \times \partial\Omega, \quad (3.39)$$

$$\hat{y}(0, \cdot) = \hat{y}_0, \hat{\theta}(0, \cdot) = \hat{\theta}_0 \quad \text{in } \Omega. \quad (3.40)$$

Assume that \hat{y} satisfies (3.28) and the temperature $\hat{\theta}$ has the following regularity

$$\begin{aligned} \hat{\theta} &\in L^\infty((0, T) \times \Omega), \\ \partial_t \hat{\theta} &\in L^2(0, T; L^\sigma(\Omega)), \quad \sigma > 1 \text{ if } N = 2, \sigma > 6/5 \text{ if } N = 3. \end{aligned} \quad (3.41)$$

In addition to condition (3.34) we assume that

$$\text{there exists } k < N, \text{ such that } n_k(x^0) \neq 0. \quad (3.42)$$

Our goal is to prove that for some $\varepsilon > 0$, whenever $(y_0, \theta_0) \in E \times L^2(\Omega)$ and

$$\|(y_0, \theta_0) - (\hat{y}_0, \hat{\theta}_0)\|_{E \times L^2(\Omega)} \leq \varepsilon, \quad (3.43)$$

we can find L^2 controls u and h with $u_k \equiv u_N \equiv 0$ such that

$$y(T, \cdot) = \hat{y}(T, \cdot) \text{ and } \theta(T, \cdot) = \hat{\theta}(T, \cdot) \text{ in } \Omega. \quad (3.44)$$

We note that for dimension $N = 2$ we are trying to control both the velocity field and the temperature by choosing the density of external heat sources in the subdomain ω . The following holds:

Theorem 3.4. *Assume that ω satisfies (3.34) and (3.42). Let $y_0 \in E$, $\theta_0 \in L^2(\Omega)$ and suppose that the pair $(\hat{y}, \hat{\theta}, \hat{p})$ solves (3.37)–(3.40) and satisfies conditions (3.28), (3.41). Then for sufficiently small $\varepsilon > 0$ there exists a solution (y, θ, p, u, h) to problem (3.36), (3.43), (3.44) such that $(u, h) \in (L^2((0, T) \times \Omega))^{N+1}$ and $u_k \equiv u_N \equiv 0$. In particular, if $N = 2$, we have local exact controllability with controls $u \equiv 0$ and $h \in L^2((0, T) \times \omega)$.*

4. Global controllability of the Navier–Stokes and the Boussinesq system

In this section we will discuss the global controllability of the Boussinesq and the Navier–Stokes systems. We start with the controllability problem for the Boussinesq system with periodic boundary conditions:

$$\partial_t y - \Delta y + (y, \nabla)y + \nabla p = f + \theta e_N + \chi_\omega u \quad \text{in } K = \prod_{j=1}^N [0, 2\pi], \quad \text{div } y = 0, \tag{4.45}$$

$$\partial_t \theta - \Delta \theta + (y, \nabla \theta) = g + \chi_\omega h \quad \text{in } K, \tag{4.46}$$

$$y(t, \dots x_i + 2\pi, \dots) = y(t, x), \quad \theta(t, \dots x_i + 2\pi, \dots) = \theta(t, x) \tag{4.47}$$

for all $i \in \{1, \dots, N\}$,

$$y(0, \cdot) = y_0, \quad \theta(0, \cdot) = \theta_0, \quad y(T, \cdot) = \hat{y}(T, \cdot), \quad \theta(T, \cdot) = \hat{\theta}(T, \cdot). \tag{4.48}$$

Here $\hat{\theta}, \hat{y}$ is some solution to the Boussinesq system with the same right-hand side:

$$\partial_t \hat{y} - \Delta \hat{y} + (\hat{y}, \nabla)\hat{y} + \nabla \hat{p} = f + \hat{\theta} e_N \quad \text{in } (0, T) \times K, \quad \text{div } \hat{y} = 0, \tag{4.49}$$

$$\partial_t \hat{\theta} - \Delta \hat{\theta} + (\hat{y}, \nabla \hat{\theta}) = g \quad \text{in } (0, T) \times K, \tag{4.50}$$

$$\hat{y}(t, \dots x_i + 2\pi, \dots) = \hat{y}(t, x), \quad \hat{\theta}(t, \dots x_i + 2\pi, \dots) = \hat{\theta}(t, x) \tag{4.51}$$

for all $i \in \{1, \dots, N\}$.

A very essential role in controllability problems for the Navier–Stokes system and the Boussinesq system is played by the type of boundary conditions.

For the case of periodic boundary conditions the situation is understood much better than for the case of Dirichlet boundary conditions. One reason for this striking difference is that for the periodic case we can construct explicitly a set of nonzero solutions of the Boussinesq system

$$\partial_t \tilde{y} - \Delta \tilde{y} + (\tilde{y}, \nabla)\tilde{y} = \nabla \tilde{p} + \tilde{\theta} e_N + \chi_\omega \tilde{u} \quad \text{in } K, \quad \text{div } \tilde{y} = 0, \quad \tilde{y}(0, \cdot) = \tilde{y}(T, \cdot) = 0, \tag{4.52}$$

$$\tilde{y}(t, \dots x_i + 2\pi, \dots) = \tilde{y}(t, x), \quad \tilde{\theta}(t, \dots x_i + 2\pi, \dots) = \tilde{\theta}(t, x) \tag{4.53}$$

for all $i \in \{1, \dots, N\}$,

$$\partial_t \tilde{\theta} - \Delta \tilde{\theta} + (\tilde{y}, \nabla \tilde{\theta}) = 0 \quad \text{in } K, \quad \tilde{\theta}(0, \cdot) = \tilde{\theta}(T, \cdot) = 0 \tag{4.54}$$

in the form

$$\tilde{y}(t, x) = m(t, x), \quad \tilde{\theta}(t, x) \equiv 0, \tag{4.55}$$

$$m(t, \dots, x_i + 2\pi, \dots) = m(t, x) \quad \text{for all } i \in \{1, \dots, N\},$$

where $m(t, x) = \nabla \gamma(t, x)$ and $\Delta \gamma(t, \cdot) = 0$ in $K \setminus \omega$ for all $t \in [0, T]$ and $\gamma(0, \cdot) = \gamma(T, \cdot) = 0$. (Obviously for the Dirichlet boundary conditions the function $\gamma \equiv 0$ is the only possible choice!) Note that for any \tilde{N} the functions $(\tilde{N}\tilde{y}, \tilde{N}\tilde{\theta}, \tilde{N}\tilde{p})$ also

solve (4.52)–(4.54) with some $\tilde{u}_{\tilde{N}}$. If we are looking for a solution of the problem (4.45), (4.46), (4.47), (4.48) in the form $(y, \theta) = (Y + \tilde{N}\tilde{m}, \theta)$ then in new equations for (Y, θ) the large parameter \tilde{N} will appear. Therefore the next logical step in finding (Y, θ) is to solve a controllability problem associated to the transport equation. In order to do that we need to make a special choice of the vector field m . The following holds:

Lemma 4.1 ([17]). *There exists a vector field $m(t, x) = (m_1(t, x), \dots, m_N(t, x)) \in C^\infty([0, T] \times K)$ such that*

$\operatorname{div} m = 0$ in $[0, T] \times K$, $m(t, x) = \nabla \gamma(t, x)$ and $\Delta \gamma = 0$ in $[0, T] \times (K \setminus \omega)$, for arbitrary $k \in \mathbb{N}$

$$m(0, x) \equiv m(T, x) \equiv 0, \quad \frac{\partial^k m(t, x)}{\partial t^k} \Big|_{t=0} = \frac{\partial^k m(t, x)}{\partial t^k} \Big|_{t=T} = 0,$$

and the relation

$$\{(t, x(t, x_0), t \in (0, T))\} \cap [0, T] \times \omega \neq \emptyset$$

is valid for every $x_0 \in K$, where $x(t, x_0)$ is solution to the Cauchy problem

$$\frac{d}{dt} x(t, x_0) = m(t, x(t, x_0)), \quad x(t, x_0) \Big|_{t=0} = x_0.$$

Moreover, $x(T, x_0) = x_0$ for each $x \in K$. Furthermore there exist a finite cover $\{\mathcal{O}_j \mid j = 1, \dots, J\}$ of K by open sets \mathcal{O}_j and a number $\hat{\delta} > 0$ such that for each j all the curves $x(t, x_0)$, $x_0 \in \mathcal{O}_j$ lie in ω for some time interval $\hat{\delta}$.

In case we choose the vector field m as in Lemma 4.1 the following controllability problem may be solved for all regular initial data y_0, θ_0 :

$$\partial_t r + (m, \nabla)r + (r, \nabla)m - \nabla q_1 = \chi_\omega \bar{u}, \quad \operatorname{div} r = 0,$$

$$\partial_t z + (m, \nabla)z = \chi_\omega \bar{h},$$

$$r(t, \dots, x_i + 2\pi, \dots) = r(t, x), \quad z(t, \dots, x_i + 2\pi, \dots) = z(t, x), \quad i \in \{1, \dots, N\},$$

$$r(0, \cdot) = y_0, \quad z(0, \cdot) = z_0, \quad r(T, \cdot) = \hat{y}(\varepsilon T, \cdot), \quad z(T, \cdot) = \hat{\theta}(\varepsilon T, \cdot).$$

Finally one can construct an approximation for the solution to problem (4.45)–(4.48) in the form

$$y(t, x) = \frac{1}{\varepsilon} m\left(\frac{t}{\varepsilon}, x\right) + r\left(\frac{t}{\varepsilon}, x\right) + y_\varepsilon, \quad \theta(t, x) = z\left(\frac{t}{\varepsilon}, x\right) + \theta_\varepsilon, \quad (4.56)$$

$$u(t, x) = \frac{1}{\varepsilon} \bar{u}\left(\frac{t}{\varepsilon}, x\right) - \chi_\omega \Delta \frac{1}{\varepsilon} m\left(\frac{t}{\varepsilon}, x\right), \quad h = \frac{1}{\varepsilon} \bar{h}\left(\frac{t}{\varepsilon}, x\right). \quad (4.57)$$

Here the terms $y_\varepsilon, \theta_\varepsilon$ are small provided that $\varepsilon > 0$ is small. Of course, we do not have the exact equality $y(\varepsilon T, \cdot) = \hat{y}(\varepsilon T, \cdot)$ but the difference $y(\varepsilon T, \cdot) - \hat{y}(\varepsilon T, \cdot)$

can be made arbitrarily small. (This proves Lions global controllability conjecture for the Boussinesq and the Navier–Stokes system with control distributed on the whole boundary.) Then the local controllability result similar to Theorem 3.4 could be applied in order to switch to the trajectory $(\hat{y}, \hat{\theta})$.

The idea to construct a solution to the controllability problem in the form (4.56), (4.57) was proposed by J. M. Coron in [3], [4] for the two dimensional Navier–Stokes system and Euler equation. In particular, Coron proved that if the control acts on an arbitrary open subset of the boundary which meets any connected component of this boundary, then the 2-D Euler equations are exactly controllable. Later the proof was extended to 3-D Euler equation by Glass [18], [19]. In [6], Coron constructed explicitly the feedback laws which globally asymptotically stabilize the fluid flow described by the Euler equation. The global exact controllability of the Navier–Stokes system on a manifold without boundary was studied in [7].

In order to formulate controllability results for the Boussinesq system rigorously we introduce the functional spaces

$$V^0(K) = \{y(x) \in (L^2(K))^N : \operatorname{div} y = 0, y(t, \dots x_i + 2\pi, \dots) = y(t, x) \text{ for all } i \in \{1, \dots, N\}\},$$

$$V^1(K) = \{y \in V^0(K) \cap (W_2^1(K))^N\},$$

$$V^{1,2} = \{y(t, x) \in (W_2^{1,2}((0, T) \times K))^N : \operatorname{div} y = 0, y(t, \dots x_i + 2\pi, \dots) = y(t, x)\}.$$

We have the following result:

Theorem 4.1 ([17]). *Let $y_0 \in V^1(K)$, $\theta_0 \in W_2^1(K)$, $f \in L^2(0, T; V^0(K))$, $g \in L^2((0, T) \times K)$ and suppose that for some $\beta \in (0, 1)$ the function $(\hat{y}, \hat{\theta}, \hat{p}) \in C^1(0, T; V^0(K) \cap (C^{2,\beta}(K))^N) \times C^1(0, T; C^{2,\beta}(K)) \times L^2(0, T; W_2^1(K))$ is a given solution of the Boussinesq system (4.49)–(4.51). Then there exists a solution $(y, \theta, p, u, h) \in V^{1,2} \times W^{1,2}((0, T) \times K) \times L^2(0, T; W_2^1(K)) \times (L^2((0, T) \times \omega))^{N+1}$ to problem (4.45), (4.46), (4.47), (4.48).*

Now we consider the problem of global controllability for the 2-D Navier–Stokes system with zero Dirichlet boundary conditions and control distributed over a part of the boundary. We need to introduce a “large parameter” in this problem but the analog of (4.55) for a general domain is hard to find. Therefore below we consider the Navier–Stokes system in the special domain $\Omega = \{(x_1, x_2) : x_1 \in (0, 1), x_2 \in (0, 1)\}$. Let us consider the following controllability problem:

$$\begin{cases} \partial_t y - \Delta y + (y, \nabla)y = \nabla p + f, & \operatorname{div} y = 0 & (t, x) \in (0, T) \times \Omega, \\ y(t, 0, x_2) = 0 & & (t, x_2) \in (0, T) \times (0, 1), \\ y(0, \cdot) = y_0, \quad y(T, \cdot) = 0 & & x = (x_1, x_2) \in \Omega. \end{cases} \quad (4.58)$$

The initial condition y_0 satisfies

$$\operatorname{div} y_0 = 0, \quad x \in \Omega, \quad \text{and} \quad y_0(0, x_2) = 0, \quad x_2 \in (0, 1). \quad (4.59)$$

Observe that in system (4.58) we did not fix traces of y on $(\{1\} \times (0, 1)) \cup ((0, 1) \times \{0, 1\})$. They can be chosen arbitrarily and considered as a boundary control.

Next we construct an analog of the vector field m . Let the function $U(t, x)$ have the form $U(t, x) = (0, z(t, x_1))$ where $z = z(t, x_1)$ solves the following problem associated to a linear heat equation:

$$\begin{cases} \partial_t z - \partial_{x_1 x_1}^2 z = c(t) & (t, x_1) \in (0, T) \times (0, 2), \\ z(t, 0) = 0, \quad z(t, 1) = w(t) & t \in (0, T), \\ z(0, x_1) = 0 & x_1 \in (0, 2). \end{cases} \quad (4.60)$$

Here $c(t)$ is a constant for each t such that

$$c(0) \neq 0, \quad w(t) \in C^\infty[0, T], \quad w(0) = 0, \quad w'(0) = c(0), \quad w''(0) = c'(0).$$

Using this function, we construct $U(t, x) = (0, z(t, x_1))$ and $q = x_2 c(t)$ for $t \in (0, T)$, $x \in \tilde{K} = [0, 1] \times [0, 2]$, which for an arbitrary $\tilde{N} \in \mathbb{R}^1$ solves

$$\begin{cases} \partial_t(\tilde{N}U) - \Delta \tilde{N}U + (\tilde{N}U, \nabla)(\tilde{N}U) = \nabla(\tilde{N}q), \\ \operatorname{div}(\tilde{N}U) = 0 & (t, x) \in (0, T) \times \tilde{K}, \\ \tilde{N}U(t, 0, x_2) = 0 & (t, x_2) \in (0, T) \times \mathbb{R}^1, \\ \tilde{N}U(0, x) = 0 & x \in \tilde{K}. \end{cases} \quad (4.61)$$

We have

Theorem 4.2 ([20]). *Let $f \in L^2((0, T) \times \Omega)$ and let $y_0 \in W_2^1(\Omega)$ satisfy (4.59). Then there exists a sequence of functions f_ε such that*

$$f_\varepsilon \rightarrow f \text{ in } L^{p_0}(0, T; V'), \quad p_0 \in (1, 8/7),$$

and there exists at least one solution to the controllability problem

$$\begin{cases} \partial_t y_\varepsilon - \Delta y_\varepsilon + (y_\varepsilon, \nabla)y_\varepsilon + \nabla p_\varepsilon = f_\varepsilon, \quad \operatorname{div} y_\varepsilon = 0 & (t, x) \in (0, T) \times \Omega, \\ y_\varepsilon(t, 0, x_2) = 0 & (t, x_2) \in (0, T) \times (0, 1), \\ y_\varepsilon(0, x) = y_0, \quad y_\varepsilon(T, x) = 0 & x \in \Omega. \end{cases} \quad (4.62)$$

The sequence of the functions y_ε can be constructed in the following way: First let us choose a sufficiently small number $\delta = \delta(\varepsilon) > 0$ such that

$$\|f\|_{L^{p_0}(T-3\delta, T; V')} \leq \varepsilon/10.$$

• On the interval between $t = 0$ and $t = T - 3\delta$, we do not exert any control. So in this interval our function y_ε is given by the solution to the Navier–Stokes system with homogeneous Dirichlet boundary conditions.

• Next, on the interval $[T - 3\delta, T - 2\delta]$, we consider a function $\tilde{y}_{0,\varepsilon} \in V \cap C_0^\infty(\Omega)$ close to $y(T - 3\delta, x)$ in V . In particular,

$$\|\tilde{y}_{0,\varepsilon} - y(T - 3\delta, \cdot)\|_V \leq \delta^3.$$

On the interval $[T - 3\delta, T - 2\delta]$ we set

$$y_\varepsilon(t, x) = \frac{(t - T + 3\delta)}{\delta} \tilde{y}_{0,\varepsilon}(x) - \frac{(t - T + 2\delta)}{\delta} y(T - 3\delta, x),$$

$$(t, x) \in [T - 3\delta, T - 2\delta] \times \Omega.$$

• As the next step, on the segment $[T - 2\delta, T - 2\delta + 2/\tilde{N}]$, we look for the solution u_ε in the form

$$y_\varepsilon(t, x) = \tilde{N}^2 \tilde{U}(t, x) + \mathbf{y}(t, x) - \tilde{V}(t, x), \quad p_\varepsilon(t, x) = \tilde{r}(t, x),$$

where $\tilde{U}(t, x) = U(t - T + 2\delta, x)$, $\mathbf{y}(t, x) = \tilde{y}(t - T + 2\delta, x)$, $\tilde{V}(t, x) = \theta(t - T + 2\delta)V(t - T + 2\delta, x)$, $\tilde{r}(t, x) = \theta(t)r(t - T + 2\delta, x)$.

The function \tilde{y} solves the following controllability problem for the transport equation:

$$\begin{cases} \partial_t \tilde{y} + \tilde{N}^2(U, \nabla) \tilde{y} + \tilde{N}^2(\tilde{y}, \nabla)U = 0 & (t, x) \in (0, T) \times \tilde{K}, \\ \tilde{y}(t, 0, x_2) = 0 & (t, x_2) \in (0, T) \times \mathbb{R}^1, \\ \tilde{y}(0, x) = \tilde{y}_{0,\varepsilon}, \quad \tilde{y}(1/\tilde{N}, x) = 0 & x \in \tilde{K}. \end{cases} \quad (4.63)$$

The function \tilde{V} is a correction term, which ensure that the vector field y_ε is divergence free:

$$\begin{cases} \partial_t V - \Delta V = \nabla r, \quad \operatorname{div} V = \operatorname{div} \tilde{y} & (t, x) \in (0, T) \times \tilde{K}, \\ V(t, 0, x_2) = V(t, 1, x_2) = 0 & (t, x_2) \in (0, T) \times \mathbb{R}^1, \\ V(t, x_1, x_2) = V(t, x_1, x_2 + 2) & (t, x_1, x_2) \in (0, T) \times (0, 2) \times \mathbb{R}^1, \\ V(0, x) = 0 & x \in \tilde{K}. \end{cases} \quad (4.64)$$

There exists a positive constant $C > 0$ independent of \tilde{N} such that

$$\|V\|_{C([0, 2/\tilde{N}]; L^2(\tilde{K}))} + \|V_{x_2}\|_{C([0, 2/\tilde{N}]; L^2(\tilde{K}))} \leq \frac{C}{\tilde{N}^{\frac{1}{8}}}. \quad (4.65)$$

This estimate is the consequence of the global version of sharp regularity result for the pressure obtained in [10]. Finally $\theta = \theta(t) \in C^2([0, 2/\tilde{N}])$ is an arbitrary function such that

$$\theta(t) = 1, \quad t \in [0, 1/\tilde{N}], \quad \text{and} \quad \theta(t) = 0 \quad \text{in a neighborhood of } 2/\tilde{N}.$$

Let $y_\varepsilon = 0$ for $(t, x) \in (T - 2\delta, T - 2\delta + 2/\tilde{N}) \times \Omega$. We set $f_\varepsilon = \partial_t y_\varepsilon - \Delta y_\varepsilon + (y_\varepsilon, \nabla) y_\varepsilon$ for all $(t, x) \in [T - 2\delta, T - 2\delta + 2/\tilde{N}] \times \Omega$.

A short computation and (4.65) imply

$$\|\tilde{f}_\varepsilon\|_{L^{p_0}(T-2\delta, T-2\delta+2/\tilde{N}; V')} \leq C\tilde{N}^{7/8-1/p_0}.$$

Thanks to our choice of p_0 , this constant tends to zero as $\tilde{N} \rightarrow +\infty$.

• Finally, on the interval $[T - 2\delta + 1/\tilde{N}, T]$, we take $f_\varepsilon \equiv 0$ and we try to find a boundary control which drives the associated solution of (4.62) which starts at time $t = T - 2\delta + 2/\tilde{N}$ from the initial condition $\tilde{N}^2 U(2/\tilde{N}, x)$ to zero at time $t = T$.

Observe that we have $y_\varepsilon(T - 2\delta + 2/\tilde{N}, x) = \tilde{N}^2 U(2/\tilde{N}, x)$ since $\theta(2/\tilde{N}) = 0$.

By Theorem 2.1 for any $\bar{z}_0 \in L^2(0, 1)$, there exists a boundary control $\rho = \rho(t) \in L^2(0, 2/\tilde{N} - 2\delta)$ such that the solution of

$$\begin{cases} \partial_t \bar{z} - \partial_{x_1 x_1}^2 \bar{z} = 0 & (t, x_1) \in (0, T) \times (0, 1), \\ \bar{z}(t, 0) = 0, \quad \bar{z}(t, 1) = \rho(t) & t \in (0, T), \\ \bar{z}(0, x_1) = \bar{z}_0 & x_1 \in (0, 1). \end{cases}$$

satisfies

$$\bar{z}(2\delta - 2/\tilde{N}, x_1) = 0, \quad x_1 \in (0, 1).$$

Then it suffices to take

$$y_\varepsilon(t, x) = (0, \bar{z}(t - T + 2\delta - 2/\tilde{N}, x_1)), \quad (t, x) \in (T - 2\delta + 2/\tilde{N}, T) \times \Omega,$$

with \bar{z} the solution of the previous null controllability problem with initial condition

$$\bar{z}_0(x_1) = z(2/\tilde{N}, x_1) \quad x_1 \in (0, 1).$$

The construction of the function y_ε is finished.

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