Port-Hamiltonian systems: an introductory survey

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Abstract. The theory of port-Hamiltonian systems provides a framework for the geometric description of network models of physical systems. It turns out that port-based network models of physical systems immediately lend themselves to a Hamiltonian description. While the usual geometric approach to Hamiltonian systems is based on the canonical symplectic structure of the phase space or on a Poisson structure that is obtained by (symmetry) reduction of the phase space, in the case of a port-Hamiltonian system the geometric structure derives from the *interconnection* of its sub-systems. This motivates to consider Dirac structures instead of Poisson structures, since this notion enables one to define Hamiltonian systems with algebraic constraints. As a result, any power-conserving interconnection of port-Hamiltonian systems again defines a port-Hamiltonian system.

The port-Hamiltonian description offers a systematic framework for analysis, control and simulation of complex physical systems, for lumped-parameter as well as for distributed-parameter models.

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1. Introduction

Historically, the Hamiltonian approach has its roots in analytical mechanics and starts from the principle of least action, and proceeds, via the Euler-Lagrange equations and the Legendre transform, towards the Hamiltonian equations of motion. On the other hand, the *network* approach stems from electrical engineering, and constitutes a cornerstone of mathematical systems theory. While most of the *analysis* of physical systems has been performed within the Lagrangian and Hamiltonian framework, the network point of view is prevailing in *modelling* and *simulation* of (complex) physical engineering systems.

The framework of port-Hamiltonian systems *combines* both points of view, by associating with the interconnection structure of the network model a *geometric structure* given by a (pseudo-) *Poisson structure*, or more generally a *Dirac structure*. The Hamiltonian dynamics is then defined with respect to this Dirac structure *and* the Hamiltonian given by the total stored energy. Furthermore, port-Hamiltonian systems are *open* dynamical systems, which interact with their environment through ports. Re-

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sistive effects are included by terminating some of these ports on energy-dissipating elements.

Dirac structures encompass the geometric structures which are classically being used in the geometrization of mechanics (that is, Poisson structures and pre-symplectic structures), and allow to describe the geometric structure of dynamical systems with algebraic *constraints*. Furthermore, Dirac structures allow to extend the Hamiltonian description of *distributed-parameter systems* to include variable boundary conditions, leading to distributed-parameter port-Hamiltonian systems with boundary ports.

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2. Finite-dimensional port-Hamiltonian systems

In this section we recapitulate the basics of finite-dimensional port-Hamiltonian systems. For more details we refer e.g. to [19], [17], [20], [33], [34], [30], [36], [12], [5].

2.1. From classical Hamiltonian equations to port-Hamiltonian systems. The standard *Hamiltonian equations* for a mechanical system are given as

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}(q, p), \\ \dot{p} &= -\frac{\partial H}{\partial q}(q, p) + F \end{aligned} \tag{1}$$

where the *Hamiltonian* $H(q, p)$ is the total energy of the system, $q = (q_1, \ldots, q_k)^T$ are generalized configuration coordinates for the mechanical system with k degrees of freedom, $p = (p_1, \ldots, p_k)^T$ is the vector of generalized momenta, and the input F is the vector of external generalized forces. The state space of (1) with local coordinates (q, p) is called the *phase space*.

One immediately derives the following *energy balance*:

$$\frac{d}{dt}H = \frac{\partial^T H}{\partial q}(q, p)\dot{q} + \frac{\partial^T H}{\partial p}(q, p)\dot{p} = \frac{\partial^T H}{\partial p}(q, p)F = \dot{q}^T F, \tag{2}$$

expressing that the increase in energy of the system is equal to the supplied work (*conservation of energy*). This motivates to define the *output* of the system as $e = \dot{q}$ (the vector of generalized velocities).

System (1) is more generally given in the following form

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}(q,p), \quad (q,p) = (q_1,\dots,q_k,p_1,\dots,p_k), \\ \dot{p} &= -\frac{\partial H}{\partial q}(q,p) + B(q)f, \qquad f \in \mathbb{R}^m, \\ e &= B^T(q)\frac{\partial H}{\partial p}(q,p) \quad (= B^T(q)\dot{q}), \quad e \in \mathbb{R}^m, \end{aligned} \tag{3}$$

with $B(q)f$ denoting the generalized forces resulting from the input $f \in \mathbb{R}^m$. In case $m < k$ we speak of an *underactuated* system. Similarly to (2) we obtain the energy balance

$$\frac{dH}{dt}(q(t),p(t)) = e^T(t)f(t). \tag{4}$$

A further generalization is to consider systems which are described in local coordinates as

$$\begin{aligned} \dot{x} &= J(x)\frac{\partial H}{\partial x}(x) + g(x)f, \qquad x \in \mathcal{X},\ f \in \mathbb{R}^m, \\ e &= g^T(x)\frac{\partial H}{\partial x}(x), \qquad e \in \mathbb{R}^m, \end{aligned} \tag{5}$$

where $J(x)$ is an $n \times n$ matrix with entries depending smoothly on x, which is assumed to be *skew-symmetric*, that is $J(x) = -J^T(x)$, and $x = (x_1,\dots,x_n)$ are local coordinates for an n-dimensional state space manifold $\mathcal{X}$ (not necessarily even-dimensional as above). Because of skew-symmetry of J we easily recover the energy-balance $\frac{dH}{dt}(x(t)) = e^T(t)f(t)$. We call (5) a *port-Hamiltonian system* with *structure matrix* $J(x)$, input matrix $g(x)$, and *Hamiltonian* H ([17], [19], [18]).

Remark 2.1. In many examples the structure matrix J will additionally satisfy an *integrability* condition (the Jacobi-identity) allowing us to find by Darboux's theorem "canonical coordinates". In this case J is the structure matrix of a *Poisson structure* on $\mathcal{X}$.

Example 2.2. An important class of systems that naturally can be written as port-Hamiltonian systems, is constituted by mechanical systems with *kinematic constraints* [22]. Consider a mechanical system locally described by k configuration variables $q = (q_1,\dots,q_k)$. Suppose that there are constraints on the generalized velocities $\dot{q}$, described as

$$A^T(q)\dot{q} = 0, \tag{6}$$

with $A(q)$ an $r \times k$ matrix of rank r everywhere. The constraints (6) are called *holonomic* if it is possible to find new configuration coordinates $\bar{q} = (\bar{q}_1,\dots,\bar{q}_k)$ such that the constraints are equivalently expressed as $\dot{\bar{q}}_{k-r+1} = \dot{\bar{q}}_{n-r+2} = \cdots = \dot{\bar{q}}_k = 0$, in which case the kinematic constraints integrate to the *geometric* constraints

$$\bar{q}_{k-r+1} = c_{k-r+1},\dots,\bar{q}_k = c_k \tag{7}$$

for certain constants $c_{k-r+1}, \ldots, c_k$ determined by the initial conditions. Then the system reduces to an *unconstrained* system in the remaining configuration coordinates $(\bar{q}_1, \ldots, \bar{q}_{k-r})$. If it is *not* possible to integrate the kinematic constraints as above, then the constraints are called *nonholonomic*. The equations of motion for the mechanical system with constraints (6) are given by the *constrained Hamiltonian equations*

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}(q, p), \\ \dot{p} &= -\frac{\partial H}{\partial q}(q, p) + A(q)\lambda + B(q) f, \\ e &= B^T(q)\frac{\partial H}{\partial p}(q, p), \\ 0 &= A^T(q)\frac{\partial H}{\partial p}(q, p). \end{aligned} \tag{8}$$

The *constrained* state space is therefore given as the following subset of the phase space:

$$\mathcal{X}_c = \left\{(q, p) \mid A^T(q)\frac{\partial H}{\partial p}(q, p) = 0\right\}. \tag{9}$$

One way of proceeding is to *eliminate* the constraint forces, and to *reduce* the equations of motion to the constrained state space, leading (see [32] for details) to a port-Hamiltonian system (5). The structure matrix of this reduced port-Hamiltonian system satisfies the Jacobi identity if and only if the constraints (6) are *holonomic* [32]. An alternative way of approaching the system (8) is to formalize it directly as an *implicit* port-Hamiltonian system (with respect to a Dirac structure), as will be the topic of Section 2.3.

2.2. From port-based network modelling to port-Hamiltonian systems. In this subsection we take a different point of view by emphasizing how port-Hamiltonian systems directly arise from *port-based network models* of physical systems.

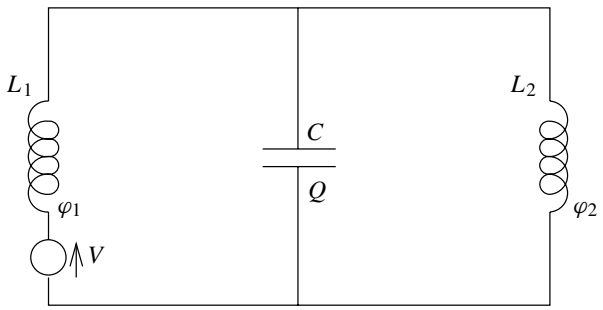

Figure 1. Controlled LC-circuit.

In network models of complex physical systems the overall system is regarded as the *interconnection* of energy-storing elements via basic interconnection (balance) laws such as Newton's third law or Kirchhoff's laws, as well as power-conserving elements like transformers, kinematic pairs and ideal constraints, together with energy-dissipating elements [3], [14], [13]. The basic point of departure for the theory of port-Hamiltonian systems is to formalize the basic interconnection laws together with the power-conserving elements by a *geometric structure*, and to define the Hamiltonian as the total energy stored in the system. This is already illustrated by the following simple example.

Example 2.3 (LCTG circuits). Consider a controlled LC-circuit (see Figure 1) consisting of two inductors with magnetic energies $H_1(\varphi_1)$, $H_2(\varphi_2)$ (φ_1 and φ_2 being the magnetic flux linkages), and a capacitor with electric energy $H_3(Q)$ (Q being the charge). If the elements are linear then $H_1(\varphi_1) = \frac{1}{2L_1}\varphi_1^2$, $H_2(\varphi_2) = \frac{1}{2L_2}\varphi_2^2$ and $H_3(Q) = \frac{1}{2C}Q^2$. Furthermore let $V = u$ denote a voltage source. Using Kirchhoff's laws one obtains the dynamical equations

$$\begin{aligned} \begin{bmatrix} \dot{Q} \\ \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{bmatrix} &= \underbrace{\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{J} \begin{bmatrix} \frac{\partial H}{\partial Q} \\ \frac{\partial H}{\partial \varphi_1} \\ \frac{\partial H}{\partial \varphi_2} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u, \\ y &= \frac{\partial H}{\partial \varphi_1} \qquad (= \text{current through voltage source}) \end{aligned} \tag{10}$$

with $H(Q, \varphi_1, \varphi_2) := H_1(\varphi_1) + H_2(\varphi_2) + H_3(Q)$ the total energy. Clearly (by Tellegen's theorem) the matrix J is skew-symmetric.

In this way every LC-circuit with independent elements can be modelled as a port-Hamiltonian system. Similarly any LCTG-circuit with independent elements can be modelled as a port-Hamiltonian system, with J now being determined by Kirchhoff's laws *and* the constitutive relations of the transformers T and gyrators G.

2.3. Dirac structures and implicit port-Hamiltonian systems. From a general modeling point of view physical systems are, at least in first instance, often described as DAE's, that is, a mixed set of differential and *algebraic* equations. This stems from the fact that in network modeling the system under consideration is regarded as obtained from interconnecting simpler sub-systems. These interconnections usually give rise to algebraic constraints between the state space variables of the sub-systems; thus leading to implicit systems. Therefore it is important to extend the framework of port-Hamiltonian systems to the context of *implicit systems*; that is, systems with algebraic constraints.

2.3.1. Dirac structures. In order to give the definition of an implicit port-Hamiltonian system we introduce the notion of a Dirac structure, formalizing the concept of

a power-conserving interconnection, and generalizing the notion of a structure matrix $J(x)$ as encountered before.

Let $\mathcal{F}$ be an ℓ-dimensional linear space, and denote its dual (the space of linear functions on $\mathcal{F}$) by $\mathcal{F}^*$. The product space $\mathcal{F} \times \mathcal{F}^*$ is considered to be the space of power variables, with power defined by

$$P = \langle f^* | f \rangle, \quad (f, f^*) \in \mathcal{F} \times \mathcal{F}^*, \tag{11}$$

where $\langle f^* | f \rangle$ denotes the duality product. Often we call $\mathcal{F}$ the space of *flows* f, and $\mathcal{F}^*$ the space of *efforts* e, with the power of an element $(f, e) \in \mathcal{F} \times \mathcal{F}^*$ denoted as $\langle e | f \rangle$.

Example 2.4. Let $\mathcal{F}$ be the space of generalized *velocities*, and $\mathcal{F}^*$ be the space of generalized *forces*, then $\langle e | f \rangle$ is mechanical power. Similarly, let $\mathcal{F}$ be the space of *currents*, and $\mathcal{F}^*$ be the space of *voltages*, then $\langle e | f \rangle$ is electrical power.

There exists on $\mathcal{F} \times \mathcal{F}^*$ the canonically defined symmetric bilinear form

$$\langle (f_1, e_1), (f_2, e_2) \rangle_{\mathcal{F} \times \mathcal{F}^*} := \langle e_1 | f_2 \rangle + \langle e_2 | f_1 \rangle \tag{12}$$

for $f_i \in \mathcal{F}$, $e_i \in \mathcal{F}^*$, $i = 1, 2$.

Definition 2.5 ([6], [8], [7]). A constant Dirac structure on $\mathcal{F}$ is a linear subspace $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$ such that

$$\mathcal{D} = \mathcal{D}^{\perp} \tag{13}$$

where $^{\perp}$ denotes the orthogonal complement with respect to the bilinear form $\langle \, , \rangle_{\mathcal{F} \times \mathcal{F}^*}$.

It immediately follows that the dimension of any Dirac structure $\mathcal{D}$ on an ℓ-dimensional linear space is equal to ℓ. Furthermore, let $(f, e) \in \mathcal{D} = \mathcal{D}^{\perp}$. Then by (12)

$$0 = \langle (f, e), (f, e) \rangle_{\mathcal{F} \times \mathcal{F}^*} = 2 \langle e \mid f \rangle. \tag{14}$$

Thus for all $(f, e) \in \mathcal{D}$ we obtain $\langle e \mid f \rangle = 0$. Hence a Dirac structure $\mathcal{D}$ on $\mathcal{F}$ defines a power-conserving relation between the power variables $(f, e) \in \mathcal{F} \times \mathcal{F}^*$, which moreover has maximal dimension.

Remark 2.6. For many systems, especially those with 3-D mechanical components, the Dirac structure is actually *modulated* by the energy or geometric variables. Furthermore, the state space $\mathcal{X}$ is a *manifold* and the flows $f_S = -\dot{x}$ corresponding to energy-storage are elements of the tangent space $T_x\mathcal{X}$ at the state $x \in \mathcal{X}$, while the efforts e_S are elements of the co-tangent space $T_x^*\mathcal{X}$.

Modulated Dirac structures often arise as a result of *kinematic constraints*. In many cases, these constraints will be configuration dependent, causing the Dirac structure to be modulated by the configuration variables, cf. Section 2.2.

In general, a port-Hamiltonian system can be represented as in Figure 2. The port variables entering the Dirac structure $\mathcal{D}$ have been split in different parts. First, there are two *internal* ports. One, denoted by $\mathcal{S}$, is corresponding to energy-storage and the other one, denoted by $\mathcal{R}$, is corresponding to internal energy-dissipation (resistive elements). Second, two *external* ports are distinguished. The external port denoted by $\mathcal{C}$ is the port that is accessible for controller action. Also the presence of *sources* may be included in this port. Finally, the external port denoted by $\mathcal{I}$ is the interaction port, defining the interaction of the system with (the rest of) its environment.

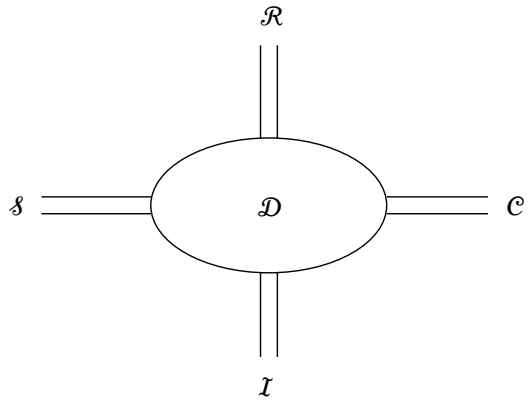

Figure 2. Port-Hamiltonian system.

2.3.2. Energy storage port. The port variables associated with the internal storage port will be denoted by (f_S, e_S). They are interconnected to the energy storage of the system which is defined by a finite-dimensional state space manifold $\mathcal{X}$ with coordinates x, together with a Hamiltonian function $H\colon \mathcal{X} \to \mathbb{R}$ denoting the energy. The flow variables of the energy storage are given by the *rate* $\dot{x}$ of the energy variables x. Furthermore, the effort variables of the energy storage are given by the *co-energy* variables $\frac{\partial H}{\partial x}(x)$, resulting in the energy balance

$$\frac{d}{dt}H = \left\langle \frac{\partial H}{\partial x}(x) \;\middle|\; \dot{x} \right\rangle = \frac{\partial^T H}{\partial x}(x)\dot{x}. \tag{15}$$

(Here we adopt the convention that $\frac{\partial H}{\partial x}(x)$ denotes the *column* vector of partial derivatives of H.)

The interconnection of the energy storing elements to the storage port of the Dirac structure is accomplished by setting

$$\begin{aligned} f_S &= -\dot{x}, \\ e_S &= \frac{\partial H}{\partial x}(x). \end{aligned} \tag{16}$$

Hence the energy balance (15) can be also written as

$$\frac{d}{dt}H = \frac{\partial^T H}{\partial x}(x)\dot{x} = -e_S^T f_S. \tag{17}$$

2.3.3. Resistive port. The second internal port corresponds to internal energy dissipation (due to friction, resistance, etc.), and its port variables are denoted by (f_R, e_R). These port variables are terminated on a static resistive relation $\mathcal{R}$. In general, a static resistive relation will be of the form

$$R(f_R, e_R) = 0, \tag{18}$$

with the property that for all (f_R, e_R) satisfying (18)

$$\langle e_R \mid f_R \rangle \leq 0. \tag{19}$$

In many cases we may restrict ourselves to *linear* resistive relations. This means that the resistive port variables (f_R, e_R) satisfy linear relations of the form

$$R_f f_R + R_e e_R = 0. \tag{20}$$

The inequality (19) corresponds to the square matrices R_f and R_e satisfying the properties of symmetry and semi-positive definiteness

$$R_f R_e^T = R_e R_f^T \geq 0, \tag{21}$$

together with the dimensionality condition $\text{rank}[R_f | R_e] = \dim f_R$.

Without the presence of additional external ports, the Dirac structure of the port-Hamiltonian system satisfies the power-balance $e_S^T f_S + e_R^T f_R = 0$ which leads to

$$\frac{d}{dt}H = -e_S^T f_S = e_R^T f_R \leq 0. \tag{22}$$

An important special case of resistive relations between f_R and e_R occurs when the resistive relations can be expressed as an *input-output* mapping $f_R = -F(e_R)$, where the resistive characteristic $F: \mathbb{R}^{m_r} \to \mathbb{R}^{m_r}$ satisfies

$$e_R^T F(e_R) \geq 0, \quad e_R \in \mathbb{R}^{m_r}. \tag{23}$$

For *linear* resistive elements this specializes to $f_R = -\tilde{R} e_R$, for some positive semi-definite symmetric matric $\tilde{R} = \tilde{R}^T \geq 0$.

2.3.4. External ports. Now, let us consider in more detail the *external* ports to the system. We distinguish between two types of external ports. One is the *control port* $\mathcal{C}$, with port variables (f_C, e_C), which are the port variables which are accessible for controller action. Other type of external port is the *interaction port* $\mathcal{I}$, which denotes the interaction of the port-Hamiltonian system with its environment. The

port variables corresponding to the interaction port are denoted by (f_I, e_I). By taking both the external ports into account the power-balance extends to

$$e_S^T f_S + e_R^T f_R + e_C^T f_C + e_I^T f_I = 0 \tag{24}$$

whereby (22) extends to

$$\frac{d}{dt}H = e_R^T f_R + e_C^T f_C + e_I^T f_I. \tag{25}$$

2.3.5. Port-Hamiltonian dynamics. The port-Hamiltonian system with state space $\mathcal{X}$, Hamiltonian H corresponding to the energy storage port $\mathcal{S}$, resistive port $\mathcal{R}$, control port $\mathcal{C}$, interconnection port $\mathcal{I}$, and total Dirac structure $\mathcal{D}$ will be succinctly denoted by $\Sigma = (\mathcal{X}, H, \mathcal{R}, \mathcal{C}, \mathcal{I}, \mathcal{D})$. The dynamics of the port-Hamiltonian system is specified by considering the constraints on the various port variables imposed by the Dirac structure, that is

$$(f_S, e_S, f_R, e_R, f_C, e_C, f_I, e_I) \in \mathcal{D},$$

and to substitute in these relations the equalities $f_S = -\dot{x}$, $e_S = \frac{\partial H}{\partial x}(x)$. This leads to the implicitly defined dynamics

$$\Big(-\dot{x}(t), \frac{\partial H}{\partial x}(x(t)), f_R(t), e_R(t), f_C, (t), e_C(t), f_I(t), e_I(t) \Big) \in \mathcal{D} \tag{26}$$

with $f_R(t), e_R(t)$ satisfying for all t the resistive relation (18):

$$R(f_R, e_R) = 0. \tag{27}$$

In many cases of interest the dynamics (26) will constrain the allowed states x, depending on the values of the external port variables (f_C, e_C) and (f_I, e_I). Thus in an equational representation port-Hamiltonian systems generally will consist of a mixed set of *differential* and *algebraic* equations (DAEs).

Example 2.7 (General LC- circuits). Consider an LC-circuit with general network topology. Kirchhoff's current and voltage laws take the general form

$$A_L{}^T I_L + A_C{}^T I_C + A_P{}^T I_P = 0,$$

$$V_L = A_L \lambda, \quad V_C = A_C \lambda, \quad V_P = A_P \lambda$$

for some matrices A_L, A_C, A_S. Here I_L, I_C, I_P denote the currents, respectively through the inductors, capacitors and external ports. Likewise, V_L, V_C, V_P denote the voltages over the inductors, capacitors and external ports. Kirchhoff's current and voltage laws define a Dirac structure between the flows and efforts:

$$f = (I_C, V_L, I_P) = (-\dot{Q}, -\dot{\phi}, I_P),$$
$$e = (V_C, I_L, V_P) = \left(\frac{\partial H}{\partial Q}, \frac{\partial H}{\partial \phi}, V_P \right)$$

with Hamiltonian $H(\phi, Q)$ the total energy. This leads the to port-Hamiltonian system in implicit form

$$\begin{aligned} -\dot{\phi} &= A_L \lambda, \\ \frac{\partial H}{\partial Q} &= A_C \lambda, \\ V_P &= A_P \lambda, \\ 0 &= {A_L}^T \frac{\partial H}{\partial \phi} - {A_C}^T \dot{Q} + {A_P}^T I_P. \end{aligned}$$

Example 2.8 (Electro-mechanical system). Consider the dynamics of an iron ball in the magnetic field of a controlled inductor: The port-Hamiltonian description of this

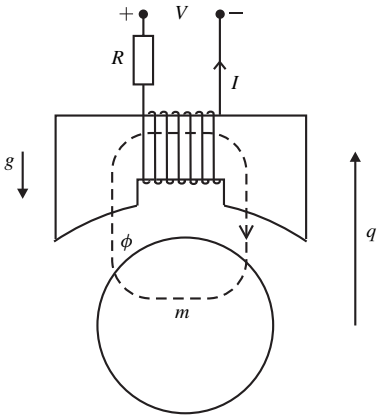

Figure 3. Magnetically levitated ball.

system (with q the height of the ball, p the vertical momentum, and φ the magnetic flux of the inductor) is given as

$$\begin{aligned} \begin{bmatrix} \dot{q} \\ \dot{p} \\ \dot{\varphi} \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -\frac{1}{R} \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \\ \frac{\partial H}{\partial \varphi} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} V, \\ I &= \frac{\partial H}{\partial \varphi}. \end{aligned} \tag{28}$$

This is a typical example of a system where the *coupling* between two different physical domains (mechanical and magnetic) takes place via the Hamiltonian

$$H(q, p, \varphi) = mgq + \frac{p^2}{2m} + \frac{\varphi^2}{2k_1(1 - \frac{q}{k_2})}$$

where the last term depends both on a magnetic variable (in this case φ) and a mechanical variable (in this case the height q).

2.4. Input-state-output port-Hamiltonian systems. An important special case of port-Hamiltonian systems is the class of *input-state-output port-Hamiltonian systems*, where there are no algebraic constraints on the state space variables, and the flow and effort variables of the resistive, control and interaction port are split into conjugated input–output pairs. Input–state–output port-Hamiltonian systems without interaction port are of the form

$$\begin{aligned} \dot{x} &= [J(x) - R(x)]\frac{\partial H}{\partial x}(x) + g(x)u, \\ y &= g^T(x)\frac{\partial H}{\partial x}(x) \end{aligned} \tag{29}$$

where u, y are the input–output pairs corresponding to the control port $\mathcal{C}$. Here the matrix $J(x)$ is skew-symmetric, while the matrix $R(x) = R^T(x) \geq 0$ specifies the resistive structure, and is given as $R(x) = g_R^T(x)\tilde{R}g_R(x)$ for some linear resistive relation $f_R = -\tilde{R}e_R$, $\tilde{R} = \tilde{R}^T \geq 0$, with g_R representing the input matrix corresponding to the resistive port. The underlying Dirac structure of the system is then given by the graph of the skew-symmetric linear map

$$\begin{pmatrix} -J(x) & -g_R(x) & -g(x) \\ g_R^T(x) & 0 & 0 \\ g^T(x) & 0 & 0 \end{pmatrix}. \tag{30}$$

3. Control by interconnection of port-Hamiltonian systems

The basic property of port-Hamiltonian systems is that the power-conserving interconnection of any number of port-Hamiltonian systems is again a port-Hamiltonian system.

To be explicit, consider two port-Hamiltonian systems Σ_A and Σ_B with Dirac structures $\mathcal{D}_A$ and $\mathcal{D}_B$ and Hamiltonians H_A and H_B, defined on state spaces $\mathcal{X}_A$, respectively $\mathcal{X}_B$. For convenience, split the ports of the Dirac structures $\mathcal{D}_A$ and $\mathcal{D}_B$ into the internal energy storage ports and all remaining external ports whose port-variables are denoted respectively by f_A, e_A and f_B, e_B. Now, consider any *interconnection* Dirac structure $\mathcal{D}_I$ involving the port-variables f_A, e_A, f_B, e_B possibly together with additional port-variables f_I, e_I. Then the interconnection of the systems Σ_A and Σ_B via $\mathcal{D}_I$ is again a port-Hamiltonian system with respect to the composed Dirac structure $\mathcal{D}_A \circ \mathcal{D}_I \circ \mathcal{D}_B$, involving as port-variables the internal storage port-variables of $\mathcal{D}_A$ and $\mathcal{D}_B$ together with the additional port-variables F_I, e_I. For details we refer to [5], [34], [30].

Furthermore, the state space of the interconnected port-Hamiltonian system is the product of the two state spaces $\mathcal{X}_A \times \mathcal{X}_B$, while its Hamiltonian is simply the sum $H_A + H_B$ of the two Hamiltonians.

This basic statement naturally extends to the interconnection of any number of port-Hamiltonian systems via an interconnection Dirac structure.

Control by port-interconnection is based on designing a controller system which is *interconnected* to the control port with port-variables (f_C, e_C). *In principle* this implies that we only consider *collocated* control, where the controller will only use the information about the plant port-Hamiltonian system that is contained in the conjugated pairs (f_C, e_C) of port variables of the control port, without using additional information about the plant (e.g. corresponding to observation on other parts of the plant system). In the second place, we will restrict attention to controller systems which are themselves *also* port-Hamiltonian systems. There are two main reasons for this. One is that by doing so the closed-loop system is *again* a port-Hamiltonian system, allowing to easily ensure some desired properties. Furthermore, it will turn out that the port-Hamiltonian framework suggests useful ways to construct port-Hamiltonian controller systems. Second reason is that port-Hamiltonian controller systems allow in principle for a physical system realization (thus linking to passive control and systems design) and physical *interpretation* of the controller action.

Since we do not know the environment (or only have very limited information about it), but on the other hand, the system *will* interact with this unknown environment, the task of the controller is often two-fold: 1) to achieve a desired control goal (e.g. set-point regulation or tracking) if the interaction with the environment is marginal or can be compensated, 2) to make sure that the controlled system has a desired interaction behavior with its environment. It is fair to say that up to now the development of the theory of control of port-Hamiltonian systems has mostly concentrated on the second aspect (which at the same time, is often underdeveloped in other control theories).

Most successful approaches to deal with the second aspect of the control goal are those based on the concept of "*passivity*", such as *dissipativity theory* [38], *impedance control* [13] and *Intrinsically Passive Control* (IPC) [36]. In fact, the port-Hamiltonian control theory can be regarded as an enhancement to the theory of passivity, making a much closer link with complex physical systems modeling at one hand and with the theory of dynamical systems (in particular, Hamiltonian dynamics) at the other hand.

As said above, we will throughout consider controller systems which are again port-Hamiltonian systems. We will use the same symbols as above for the internal and external ports and port-variables of the controller port-Hamiltonian system, with an added overbar $\bar{}$ or a superscript ^{c} in order to distinguish it from the plant system. (The interaction port of the controller system may be thought of as an extra possibility for additional controller action (outer-loop control).) In order to further distinguish the plant system and the controller we denote the state space of the plant system by $\mathcal{X}_p$ with coordinates x_p, the Dirac structure by $\mathcal{D}_p$ and its Hamiltonian by H_p, while we will denote the state space manifold of the controller system by $\mathcal{X}_c$ with coordinates x_c, its Dirac structure by $\mathcal{D}_c$ and its Hamiltonian by $H_c \colon \mathcal{X}_c \to \mathbb{R}$. The interconnection

of the plant port-Hamiltonian system with the controller port-Hamiltonian system is obtained by equalizing the port variables at the control port by

$$\begin{aligned} f_C &= -\bar{f}_C, \\ e_C &= \bar{e}_C \end{aligned} \tag{31}$$

where $\bar{f}_C$, $\bar{e}_C$ denote the control port variables of the controller system. Here, the minus sign is inserted to have a uniform notion of direction of power flow. Clearly, this 'synchronizing' interconnection is power-conserving, that is $e_C^T f_C + \bar{e}_C^T \bar{f}_C = 0$.

Remark 3.1. A sometimes useful alternative is the *gyrating* power-conserving interconnection

$$\begin{aligned} f_C &= -\bar{e}_C, \\ e_C &= \bar{f}_C. \end{aligned} \tag{32}$$

In fact, the standard feedback interconnection can be regarded to be of this type.

For both interconnection constraints it directly follows from the theory of composition of Dirac structures that the interconnected (closed-loop) system is again a port-Hamiltonian system with Dirac structure determined by the Dirac structures of the plant PH system and the controller PH system.

The resulting interconnected PH system has state space $\mathcal{X}_p \times \mathcal{X}_c$, Hamiltonian $H_p + H_c$, resistive ports $(f_R, e_R, \bar{f}_R, \bar{e}_R)$ and interaction ports $(f_I, e_I, \bar{f}_I, \bar{e}_I)$, satisfying the power-balance

$$\frac{d}{dt}(H_p + H_c) = e_R^T f_R + \bar{e}_R^T \bar{f}_R + e_I^T f_I + \bar{e}_I^T \bar{f}_I \leq e_I^T f_I + \bar{e}_I^T \bar{f}_I \tag{33}$$

since both $e_R^T f_R \leq 0$ and $\bar{e}_R^T \bar{f}_R \leq 0$. Hence we immediately recover the state space formulation of the passivity theorem, see e.g. [31], if H_p and H_c are both non-negative, implying that the plant and the controller system are passive (with respect to their controller and interaction ports and storage functions H_p and H_c), then also the closed -loop system is passive (with respect to the interaction ports and storage function $H_p + H_c$.)

Furthermore, due to the Hamiltonian structure, we can go *beyond* the passivity theorem, and we can derive conditions which ensure that we can passify and/or stabilize plant port-Hamiltonian systems for which the Hamiltonian H_p does *not* have a minimum at the desired equilibrium.

3.1. Stabilization by Casimir generation. What does the power-balance (33)mean for the stability properties of the closed-loop system, and how can we design the controller port-Hamiltonian system in such a way that the closed-loop system has desired stability properties? Let us first consider the stability of an arbitrary port-Hamiltonian system $\Sigma = (\mathcal{X}, H, \mathcal{R}, \mathcal{C}, \mathcal{I}, \mathcal{D})$ *without* control or interaction ports,

that is, an autonomous port-Hamiltonian system $\Sigma = (\mathcal{X}, H, \mathcal{R}, \mathcal{D})$. Clearly, the power-balance (33) reduces to

$$\frac{d}{dt}H = e_R^T f_R \leq 0. \tag{34}$$

Hence we immediately infer by standard Lyapunov theory that if x^* is a minimum of the Hamiltonian H then it will be a *stable* equilibrium of the autonomous port-Hamiltonian system $\Sigma = (\mathcal{X}, H, \mathcal{R}, \mathcal{D})$, which is actually *asymptotically stable* if the dissipation term $e_R^T f_R$ is negative *definite* outside x^*, or alternatively if some sort of detectability condition is satisfied, guaranteeing asymptotic stability by the use of LaSalle's Invariance principle (see for details e.g. [31]).

However, what can we say if x^* is *not* a minimum of H, and thus we cannot directly use H as a Lyapunov function?

A well-known method in Hamiltonian systems, sometimes called the Energy-Casimir method, is to use in the Lyapunov analysis next to the Hamiltonian *other* conserved quantities (dynamical invariants) which may be present in the system. Indeed, if we may find other conserved quantities then candidate Lyapunov functions can be sought within the class of *combinations* of the Hamiltonian H and those conserved quantities. In particular, if we can find a conserved quantity $C\colon \mathcal{X} \to \mathbb{R}$ such that $V := H + C$ has a minimum at the desired equilibrium x^* then we can still infer stability or asymptotic stability by replacing (34) by

$$\frac{d}{dt}V = e_R^T f_R \leq 0 \tag{35}$$

and thus using V as a Lyapunov function.

For the application of the Energy-Casimir method one may distinguish between two main cases. First situation occurs if the desired equilibrium x^* is not a stationary point of H, and one looks for a conserved quantity C such that $H + C$ has a minimum at x^*. This for example happens in the case that the desired set-point x^* is *not* an equilibrium of the uncontrolled system, but only a controlled equilibrium of the system. Second situation occurs when x^* *is* a stationary point of H, but not a minimum.

Functions that are conserved quantities of the system for *every* Hamiltonian are called *Casimir functions* or simply Casimirs. Casimirs are completely characterized by the Dirac structure of the port-Hamiltonian system. Indeed, a function $C\colon \mathcal{X} \to \mathbb{R}$ is a Casimir function of the autonomous port-Hamiltonian system (without energy dissipation) $\Sigma = (\mathcal{X}, H, \mathcal{D})$ if and only if the gradient vector $e = \frac{\partial^T C}{\partial x}$ satisfies

$$e^T f_S = 0 \quad \text{for all } f_S \text{ for which there exists } e_S \text{ such that } (f_S, e_S) \in \mathcal{D}. \tag{36}$$

Indeed, (36) is equivalent to

$$\frac{d}{dt}C = \frac{\partial^T C}{\partial x}(x(t))\dot{x}(t) = \frac{\partial^T C}{\partial x}(x(t))\, f_S = e^T f_S = 0 \tag{37}$$

for every port-Hamiltonian system $(\mathcal{X}, H, \mathcal{D})$ with the same Dirac structure $\mathcal{D}$. By the generalized skew-symmetry of the Dirac structure (36) is equivalent to the requirement that $e = \frac{\partial^T C}{\partial x}$ satisfies

$$(0, e) \in \mathcal{D}.$$

Similarly, we define a Casimir function for a port-Hamiltonian system with dissipation $\Sigma = (\mathcal{X}, H, \mathcal{R}, \mathcal{D})$ to be any function $C\colon \mathcal{X} \to \mathbb{R}$ satisfying

$$(0, e, 0, 0) \in \mathcal{D}. \tag{38}$$

Indeed, this will imply that

$$\frac{d}{dt}C = \frac{\partial^T C}{\partial x}(x(t))\dot{x}(t) = \frac{\partial^T C}{\partial x}(x(t))\, f_p = e^T f_p = 0 \tag{39}$$

for every port-Hamiltonian system $(\mathcal{X}, H, \mathcal{R}, \mathcal{D})$ with the same Dirac structure $\mathcal{D}$. (In fact by definiteness of the resistive structures the satisfaction of (39) for a particular resistive structure $\mathcal{R}$ *implies* the satisfaction for *all* resistive structures $\mathcal{R}$.)

Now let us come back to the design of a controller port-Hamiltonian system such that the closed-loop system has desired stability properties. Suppose we want to stabilize the plant port-Hamiltonian system $(\mathcal{X}_p, H_p, \mathcal{R}, \mathcal{C}, \mathcal{D}_p)$ around a desired equilibrium x_p^*. We know that for every controller port-Hamiltonian system the closed-loop system satisfies

$$\frac{d}{dt}(H_p + H_c) = e_R^T f_R + \bar{e}_R^T \bar{f}_R \leq 0. \tag{40}$$

What if x^* is not a minimum for H_p? A possible strategy is to *generate* Casimir functions $C(x_p, x_c)$ for the closed-loop system by choosing the controller port-Hamiltonian system in an appropriate way. Thereby we generate candidate Lyapunov functions for the closed-loop system of the form

$$V(x_p, x_c) := H_p(x_p) + H_c(x_c) + C(x_p, x_c)$$

where the controller Hamiltonian function $H_c\colon \mathcal{X}_c \to \mathbb{R}$ still has to be designed. The goal is thus to construct a function V as above in such a way that V has a minimum at (x_p^*, x_c^*) where x_c^* still remains to be chosen. This strategy thus is based on finding all the achievable closed-loop Casimirs. Furthermore, since the closed-loop Casimirs are based on the closed-loop Dirac structures, this reduces to finding all the achievable closed-loop Dirac structures $\mathcal{D} \circ \bar{\mathcal{D}}$.

Another way to interpret the generation of Casimirs for the closed-loop system is to look at the level sets of the Casimirs as *invariant submanifolds* of the combined plant and controller state space $\mathcal{X}_p \times \mathcal{X}_c$. Restricted to every such invariant submanifold (part of) the controller state can be expressed as a function of the plant state, whence the closed-loop Hamiltonian restricted to such an invariant manifold can be seen as a

shaped version of the plant Hamiltonian. To be explicit (see e.g. [31], [24], [25] for details) suppose that we have found Casimirs of the form

$$x_{ci} - F_i(x_p), \quad i = 1, \ldots, n_p$$

where n_p is the dimension of the controller state space, then on every invariant manifold $x_{ci} - F_i(x_p) = \alpha_i$, $i = 1, \ldots, n_p$, where $\alpha = (\alpha_1, \ldots, \alpha_{n_p})$ is a vector of constants depending on the initial plant and controller state, the closed-loop Hamiltonian can be written as

$$H_s(x_p) := H_p(x_p) + H_c(F(x_p) + \alpha),$$

where, as before, the controller Hamiltonian H_c still can be assigned. This can be regarded as *shaping* the original plant Hamiltonian H_p to a new Hamiltonian H_s.

3.2. Port Control. In broad terms, the *Port Control* problem is to design, given the plant port-Hamiltonian system, a controller port-Hamiltonian system such that the *behavior* at the interaction port of the plant port-Hamiltonian system is a desired one, or close to a desired one. This means that by adding the controller system we seek to shape the external behavior at the interaction port of the plant system. If the desired external behavior at this interaction port is given in input–output form as a desired (dynamic) impedance, then this amounts to the Impedance Control problem as introduced and studied by Hogan and co-workers [13]; see also [36] for subsequent developments.

The Port Control problem, as stated in this generality, immediately leads to two fundamental questions: 1). Given the plant PH system, and the controller PH system to be arbitrarily designed, what are the achievable behaviors of the closed-loop system at the interaction port of the plant? 2). If the desired behavior at the interaction port of the plant is not achievable, then what is the closest achievable behavior? Of course, the second question leaves much room for interpretation, since there is no obvious interpretation of what we mean by 'closest behavior'. Also the first question in its full generality is not easy to answer, and we shall only address an important subproblem.

An obvious observation is that the desired behavior, in order to be achievable, needs to be the port behavior of a PH system. This leads already to the problem of characterizing those external behaviors which are port behaviors of port-Hamiltonian systems. Secondly, the Port Control problem can be split into a number of subproblems. Indeed, we know that the closed-loop system arising from interconnection of the plant PH system with the controller PH system is specified by a Hamiltonian which is just the sum of the plant Hamiltonian and the controller Hamiltonian, and a resistive structure which is the "product" of the resistive structure of the plant and of the controller system, together with a Dirac structure which is the *composition* of the plant Dirac structure and the controller Dirac structure. Therefore an important subproblem is again to characterize the *achievable closed-loop Dirac structures.* On the other hand, a fundamental problem in addressing the Port Control problem in general theoretical terms is the lack of a systematic way to specify 'desired behavior'.

The problem of Port Control is to determine the controller system in such a way that the port behavior in the port variables f_I, e_I is a desired one. In this particular (simple and linear) example the desired behavior can be quantified e.g. in terms of a desired stiffness and damping of the closed-loop system, which is easily expressed in terms of the closed-loop transfer function from f_I to e_I. Of course, on top of the requirements on the closed-loop transfer function we would also require internal stability of the closed-loop system. For an appealing example of port control of port-Hamiltonian systems within a context of hydraulic systems we refer to [15].

3.3. Energy Control. Consider two port-Hamiltonian systems Σ_i (without internal dissipation) in input–state–output form

$$\begin{aligned} \dot{x}_i &= J_i(x_i)\frac{\partial H_i}{\partial x_i} + g_i(x_i)u_i, \\ y_i &= g_i^T(x_i)\frac{\partial H_i}{\partial x_i}, \qquad i = 1, 2, \end{aligned} \tag{41}$$

both satisfying the power-balance $\frac{d}{dt}H_i = y_i^T u_i$. Suppose now that we want to transfer the energy from the port-Hamiltonian system Σ_1 to the port-Hamiltonian system Σ_2, while keeping the total energy $H_1 + H_2$ constant. This can be done by using the following output feedback

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 & -y_1 y_2^T \\ y_2 y_1^T & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}. \tag{42}$$

Since the matrix in (42) is skew-symmetric it immediately follows that the closed-loop system composed of systems Σ_1 and Σ_2 linked by the power-conserving feedback is energy-preserving, that is $\frac{d}{dt}(H_1 + H_2) = 0$. However, if we consider the individual energies then we notice that

$$\frac{d}{dt}H_1 = -y_1^T y_1 y_2^T y_2 = -\|y_1\|^2\|y_2\|^2 \leq 0 \tag{43}$$

implying that H_1 is decreasing as long as $\|y_1\|$ and $\|y_2\|$ are different from 0. Conversely, as expected since the total energy is constant,

$$\frac{d}{dt}H_2 = y_2^T y_2 y_1^T y_1 = \|y_2\|^2\|y_1\|^2 \geq 0 \tag{44}$$

implying that H_2 is increasing at the same rate. In particular, if H_1 has a minimum at the zero equilibrium, and Σ_1 is zero-state observable, then all the energy H_1 of Σ_1 will be transferred to Σ_2, provided that $\|y_2\|$ is not identically zero (which again can be guaranteed by assuming that H_2 has a minimum at the zero equilibrium, and that Σ_2 is zero-state observable).

If there is internal energy dissipation, then this energy transfer mechanism still works. However, the fact that H_2 grows or not will depend on the balance between the energy delivered by Σ_1 to Σ_2 and the internal loss of energy in Σ_2 due to dissipation.

We conclude that this particular scheme of power-conserving energy transfer is accomplished by a skew-symmetric output feedback, which is *modulated* by the values of the output vectors of both systems. Of course this raises, among others, the question of the efficiency of the proposed energy-transfer scheme, and the need for a systematic quest of similar power-conserving energy-transfer schemes. We refer to [9] for a similar but different energy-transfer scheme directly motivated by the structure of the example (control of a snakeboard).

3.4. Achievable closed-loop Dirac structures. In all the control problems discussed above the basic question comes up what are the achievable closed-loop Dirac structures based on a given plant Dirac structure and a controller Dirac structure, which still is to be determined.

Theorem 3.2 ([5]). *Given any plant Dirac structure $\mathcal{D}_p$, a certain interconnected $\mathcal{D} = \mathcal{D}_p \circ \mathcal{D}_c$ can be achieved by a proper choice of the controller Dirac structure $\mathcal{D}_c$ if and only if the following two equivalent conditions are satisfied:*

$$\mathcal{D}_p^0 \subset \mathcal{D}^0,$$
$$\mathcal{D}^\pi \subset \mathcal{D}_p^\pi$$

where

$$\mathcal{D}_p^0 := \{f_1, e_1) \mid (f_1, e_1, 0, 0) \in \mathcal{D}_p\},$$
$$\mathcal{D}_p^\pi := \{(f_1, e_1) \mid \textit{there exists } (f_2^P, e_2^P) \textit{ with } (f_1, e_1, f_2^P, e_2^P) \in \mathcal{D}_p\},$$
$$\mathcal{D}^0 := \{(f_1, e_1) \mid (f_1, e_1, 0, 0) \in \mathcal{D}\},$$
$$\mathcal{D}^\pi := \{(f_1, e_1) \mid \textit{there exists } (f_3, e_3) \textit{ with } (f_1, e_1, f_3, e_3) \in \mathcal{D}\}.$$

An important application of the above theorem concerns the characterization of Casimir functions which can be achieved by interconnecting a given plant port-Hamiltonian system with a controller port-Hamiltonian system.

4. Distributed-parameter port-Hamiltonian systems

The treatment of infinite-dimensional Hamiltonian systems in the literature is mostly confined to systems with boundary conditions such that the energy exchange through the boundary is *zero*. On the other hand, in many applications the interaction with the environment (e.g. actuation or measurement) will actually take place through the boundary of the system. In [35] a framework has been developed to represent classes of physical distributed-parameter systems with boundary energy flow as *infinite-dimensional port-Hamiltonian systems*. It turns out that in order to allow the inclusion of boundary variables in distributed-parameter systems the concept of (an

infinite-dimensional) Dirac structure provides again the right type of generalization with respect to the existing framework [23] using Poisson structures.

As we will discuss in the next three examples, the port-Hamiltonian formulation of distributed-parameter systems is closely related to the general framework for describing basic distributed-parameter systems as systems of conservation laws, see e.g. [11], [37].

Example 4.1 (Inviscid Burger's equation). The viscous *Burger's equation* is a scalar parabolic equation defined on a one-dimensional spatial domain (interval) $Z = [a, b] \subset \mathbb{R}$, with the state variable $\alpha(t, z) \in \mathbb{R}$, $z \in Z$, $t \in I$, where I is an interval of $\mathbb{R}$, satisfying the partial differential equation

$$\frac{\partial \alpha}{\partial t} + \alpha \frac{\partial \alpha}{\partial z} - \nu \frac{\partial^2 \alpha}{\partial z^2} = 0. \tag{45}$$

The *inviscid* ($\nu = 0$) Burger's equations may be alternatively expressed as

$$\frac{\partial \alpha}{\partial t} + \frac{\partial}{\partial z} \beta = 0 \tag{46}$$

where the state variable $\alpha(t, z)$ is called the *conserved quantity* and the function $\beta := \frac{\alpha^2}{2}$ the *flux* variable. Eq. (46) is called a *conservation law*, since by integration one obtains the *balance equation*

$$\frac{d}{dt} \int_a^b \alpha \, dz = \beta(a) - \beta(b). \tag{47}$$

Furthermore, according to the framework of Irreversible Thermodynamics [27], one may express the flux β as a function of the *generating force* which is the *variational derivative* of some functional $H(\alpha)$ of the state variable. The variational derivative $\frac{\delta H}{\delta \alpha}$ of a functional $H(\alpha)$ is uniquely defined by the requirement

$$H(\alpha + \varepsilon \eta) = H(\alpha) + \varepsilon \int_a^b \frac{\delta H}{\delta \alpha} \eta \, dz + O(\varepsilon^2) \tag{48}$$

for any $\varepsilon \in \mathbb{R}$ and any smooth function $\eta(z, t)$ such that $\alpha + \varepsilon \eta$ satisfies the same boundary conditions as α [23]. For the inviscid Burger's equation one has $\beta = \frac{\delta H}{\delta \alpha}$, where

$$H(\alpha) = \int_a^b \frac{\alpha^3}{6} \, dz. \tag{49}$$

Hence the inviscid Burger's equation may be also expressed as

$$\frac{\partial \alpha}{\partial t} = -\frac{\partial}{\partial z} \frac{\delta H}{\delta \alpha}. \tag{50}$$

This defines an infinite-dimensional Hamiltonian system in the sense of [23] with respect to the skew-symmetric operator $\frac{\partial}{\partial z}$ that is defined on the functions with support contained in the interior of the interval Z.

From this formulation one derives that the Hamiltonian $H(\alpha)$ is *another* conserved quantity. Indeed, by integration by parts

$$\frac{d}{dt}H = \int_a^b \frac{\delta H}{\delta \alpha} \cdot -\frac{\partial}{\partial z}\frac{\delta H}{\delta \alpha} dz = \frac{1}{2}\left(\beta^2(a) - \beta^2(b)\right). \tag{51}$$

We note that the right-hand side is a *function of the flux variables* evaluated at the boundary of the spatial domain Z.

The second example consists of a system of *two* conservations laws, corresponding to the case of two physical domains in interaction.

Example 4.2 (The p-system, cf. [11], [37]). The p-system is a model for e.g. a one-dimensional gas dynamics. Again, the spatial variable z belongs to an interval $Z \subset \mathbb{R}$, while the dependent variables are the specific volume $v(t, z) \in \mathbb{R}^+$, the velocity $u(t, z)$ and the pressure functional $p(v)$ (which for instance in the case of an ideal gas with constant entropy is given by $p(v) = Av^{-\gamma}$ where $\gamma \geq 1$). The *p-system* is then defined by the following system of partial differential equations

$$\begin{aligned} \frac{\partial v}{\partial t} - \frac{\partial u}{\partial z} &= 0, \\ \frac{\partial u}{\partial t} + \frac{\partial\, p(v)}{\partial z} &= 0 \end{aligned} \tag{52}$$

representing respectively conservation of mass and of momentum. By defining the state vector as $\alpha(t, z) = (v, u)^T$, and the vector-valued flux $\beta(t, z) = (-u, p(v))^T$ the p-system is rewritten as

$$\frac{\partial \alpha}{\partial t} + \frac{\partial}{\partial z}\beta = 0. \tag{53}$$

Again, according to the framework of Irreversible Thermodynamics, the flux vector may be written as function of the variational derivatives of some functional. Indeed, consider the energy functional $H(\alpha) = \int_a^b \mathcal{H}(v, u)dz$ where the energy density $\mathcal{H}(v, u)$ is given as the sum of the internal energy and the kinetic energy densities

$$\mathcal{H}(v, u) = \mathcal{U}(v) + \frac{u^2}{2} \tag{54}$$

with $-\mathcal{U}(v)$ a primitive function of the pressure. (Note that for simplicity the mass density has been set equal to 1, and hence no difference is made between the velocity and the momentum.) The flux vector β may be expressed in terms of the variational derivatives of H as

$$\beta = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta v} \\ \frac{\delta H}{\delta u} \end{pmatrix}. \tag{55}$$

The anti-diagonal matrix represents the canonical coupling between two physical domains: the kinetic and the potential (internal) domain. Thus the variational derivative

of the total energy with respect to the state variable of one domain generates the flux variable for the other domain. Combining eqns. (53) and (55), the p-system may thus be written as the Hamiltonian system

$$\begin{pmatrix} \frac{\partial \alpha_1}{\partial t} \\ \frac{\partial \alpha_2}{\partial t} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{\partial}{\partial z} \\ -\frac{\partial}{\partial z} & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta \alpha_1} \\ \frac{\delta H}{\delta \alpha_2} \end{pmatrix}. \tag{56}$$

Using again integration by parts, one may derive the following *energy balance equation*:

$$\frac{d}{dt} H = \beta_1(a)\beta_2(a) - \beta_1(b)\beta_2(b). \tag{57}$$

Notice again that the right-hand side of this power-balance equation is a quadratic function of the fluxes at the boundary of the spatial domain.

The last example is the *vibrating string*. It is again a system of two conservation laws representing the canonical interdomain coupling between the kinetic energy and the elastic potential energy. However in this example the *classical* choice of the state variables leads to express the total energy as a function of some of the *spatial derivatives* of the state variables.

Example 4.3 (Vibrating string). Consider an elastic string subject to traction forces at its ends, with spatial variable $z \in Z = [a, b] \subset \mathbb{R}$. Denote by $u(t, z)$ the displacement of the string and the velocity by $v(t, z) = \frac{\partial u}{\partial t}$. Using the vector of state variables $x(t, z) = (u, v)^T$, the dynamics of the vibrating string is described by the system of partial differential equations

$$\frac{\partial x}{\partial t} = \begin{pmatrix} v \\ \frac{1}{\mu}\frac{\partial}{\partial z}\left(T\frac{\partial u}{\partial z}\right) \end{pmatrix} \tag{58}$$

where the first equation is simply the definition of the velocity and the second one is Newton's second law. Here T denotes the elasticity modulus, and μ the mass density. The total energy is $H(x) = U(u) + K(v)$, where the elastic potential energy U is a function of the *strain* $\frac{\partial u}{\partial z}(t, z)$

$$U(u) = \int_a^b \frac{1}{2}\, T \left(\frac{\partial u}{\partial z}\right)^2 dz \tag{59}$$

and the kinetic energy K depends on the velocity $v(t, z) = \frac{\partial u}{\partial t}$ as

$$K(v) = \int_a^b \frac{1}{2}\, \mu v(t, z)^2\, dz. \tag{60}$$

Thus the total system (58) may be expressed as

$$\frac{\partial x}{\partial t} = \begin{pmatrix} 0 & \frac{1}{\mu} \\ -\frac{1}{\mu} & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta u} \\ \frac{\delta H}{\delta v} \end{pmatrix} \tag{61}$$

where $\frac{\delta H}{\delta u} = \frac{\delta U}{\delta u} = -\frac{\partial}{\partial z}\left(T\,\frac{\partial u}{\partial z}\right)$ is the elastic force and $\frac{\delta H}{\delta v} = \frac{\delta K}{\delta v} = \mu v$ is the momentum.

In this formulation, the system is *not* anymore expressed as a system of conservation laws since the time-derivative of the state variables is a function of the variational derivatives of the energy *directly*, and *not* the spatial derivative of a function of the variational derivatives as before. Instead of being a simplification, this reveals a drawback for the case of non-zero energy flow through the boundary of the spatial domain. Indeed, in this case the *variational derivative has to be completed by a boundary term* since the Hamiltonian functional depends on the *spatial derivatives of the state*. For example, in the computation of the variational derivative of the elastic potential energy U one obtains by integration by parts that $U(u+\varepsilon\eta) - U(u)$ equals

$$-\varepsilon \int_a^b \frac{\partial}{\partial z}\left(T\frac{\partial u}{\partial z}\right)\eta\, dz + \varepsilon\left[\eta\left(T\frac{\partial u}{\partial z}\right)\right]_a^b + O(\varepsilon^2) \tag{62}$$

and the second term in this expression constitutes an extra boundary term.

Alternatively we now formulate the vibrating string as a system of two conservation laws. Take as alternative vector of state variables $\alpha(t,z) = (\varepsilon,\, p)^T$, where ε denotes the *strain* $\alpha_1 = \varepsilon = \frac{\partial u}{\partial z}$ and p denotes the *momentum* $\alpha_2 = p = \mu v$. Recall that in these variables the total energy is written as

$$H_0 = \int_a^b \frac{1}{2}\left(T\alpha_1^2 + \frac{1}{\mu}\alpha_2^2\right) dz \tag{63}$$

and directly depends on the state variables and *not* on their spatial derivatives. Furthermore, one defines the flux variables to be the *stress* $\beta_1 = \frac{\delta H_0}{\delta \alpha_1} = T\alpha_1$ and the *velocity* $\beta_2 = \frac{\delta H_0}{\delta \alpha_1} = \frac{\alpha_2}{\mu}$. In matrix notation, the flux vector β is thus expressed as a function of the variational derivatives $\frac{\delta H_0}{\delta \alpha}$ by

$$\beta = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \frac{\delta H_0}{\delta \alpha}. \tag{64}$$

Hence the vibrating string may be alternatively expressed by the system of two conservation laws

$$\frac{\partial \alpha}{\partial t} = \begin{pmatrix} 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 \end{pmatrix} \frac{\delta H_0}{\delta \alpha} \tag{65}$$

satisfying the power balance equation (57).

4.1. Systems of two conservation laws in interaction. Let us now consider the *general class* of distributed-parameter systems consisting of two conservation laws with the canonical coupling as in the above examples of the p-system and the vibrating string. Let the spatial domain $Z \subset \mathbb{R}^n$ be an n-dimensional smooth manifold with smooth $(n-1)$-dimensional boundary ∂Z. Denote by $\Omega^k(Z)$ the vector space of (differential) k-forms on Z (respectively by $\Omega^k(\partial Z)$ the vector space of k-forms on ∂Z).

Denote furthermore by $\Omega = \bigoplus_{k\geq 0} \Omega^k(Z)$ the algebra of differential forms over Z and recall that it is endowed with an exterior product $\wedge$ and an exterior derivation d.

Definition 4.4. A *system of conservation laws* is defined by a set of *conserved quantities* $\alpha_i \in \Omega^{k_i}(Z)$, $i \in \{1, \ldots, N\}$ where $N \in \mathbb{N}, k_i \in \mathbb{N}$, defining the state space $\mathcal{X} = \bigotimes_{i=1,\ldots,N} \Omega^{k_i}(Z)$, and satisfying a set of *conservation laws*

$$\frac{\partial \alpha_i}{\partial t} + \mathrm{d}\beta_i = g_i \tag{66}$$

where $\beta_i \in \Omega^{k_i-1}(Z)$ denote the set of *fluxes* and $g_i \in \Omega^{k_i}(Z)$ denote the set of *distributed interaction forms*. In general, the fluxes β_i are defined by so-called *closure equations*

$$\beta_i = J(\alpha_i, z), \quad i = 1, \ldots, N \tag{67}$$

leading to a closed form for the dynamics of the conserved quantities α_i. The integral form of the conservation laws yields the following *balance equations*

$$\frac{d}{dt}\int_S \alpha_i + \int_{\partial S} \beta_i = \int_S g_i \tag{68}$$

for any surface $S \subset Z$ of dimension equal to the degree of α_i.

Remark 4.5. A common case is that $Z = \mathbb{R}^3$ and that the conserved quantities are 3-forms, that is, the balance equation is evaluated on volumes of the 3-dimensional space. In this case () takes in vector calculus notation the familiar form

$$\frac{\partial \alpha_i}{\partial t}(z,t) + \mathrm{div}_z \beta_i = g_i, \quad i = 1, \ldots, n. \tag{69}$$

However, systems of conservation laws may correspond to differential forms of any degree. Maxwell's equations are an example where the conserved quantities are differential forms of degree 2.

In the sequel, as in our examples sofar, we consider a particular class of systems of conservation laws where the closure equations are such that fluxes are linear functions of the variational derivatives of the Hamiltonian functional. First recall the general definition of the *variational derivative* of a functional $H(\alpha)$ with respect to the differential form $\alpha \in \Omega^p(Z)$ (generalizing the definition given before).

Definition 4.6. Consider a density function $\mathcal{H}\colon \Omega^p(Z) \times Z \to \Omega^n(Z)$ where $p \in \{1, \ldots, n\}$, and denote by $H := \int_Z \mathcal{H} \in \mathbb{R}$ the associated functional. Then the uniquely defined differential form $\frac{\delta H}{\delta \alpha} \in \Omega^{n-p}(Z)$ which satisfies for all $\Delta\alpha \in \Omega^p(Z)$ and $\varepsilon \in \mathbb{R}$

$$H(\alpha + \varepsilon\Delta\alpha) = \int_Z \mathcal{H}(\alpha) + \varepsilon \int_Z \left[\frac{\delta H}{\delta \alpha} \wedge \Delta\alpha\right] + \mathrm{O}\left(\varepsilon^2\right)$$

is called the *variational derivative* of H with respect to $\alpha \in \Omega^p(Z)$.

Definition 4.7. *Systems of two conservation laws with canonical interdomain coupling* are systems involving a pair of conserved quantities $\alpha_p \in \Omega^p(Z)$ and $\alpha_q \in \Omega^q(Z)$, differential forms on the n-dimensional spatial domain Z of degree p and q respectively, satisfying $p+q = n+1$ ('complementarity of degrees'). The closure equations generated by a *Hamiltonian density function* $\mathcal{H}: \Omega^p(Z) \times \Omega^q(Z) \times Z \to \Omega^n(Z)$ resulting in the Hamiltonian $H := \int_Z \mathcal{H} \in \mathbb{R}$ are given by

$$\begin{pmatrix} \beta_p \\ \beta_q \end{pmatrix} = \varepsilon \begin{pmatrix} 0 & (-1)^r \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta \alpha_p} \\ \frac{\delta H}{\delta \alpha_q} \end{pmatrix} \tag{70}$$

where $r = pq+1$, $\varepsilon \in \{-1, +1\}$, depending on the sign convention of the considered physical domain.

Define the vector of *flow variables* to be the time-variation of the state, and the *effort variables* to be the variational derivatives

$$\begin{pmatrix} f_p \\ f_q \end{pmatrix} = \begin{pmatrix} \frac{\partial \alpha_p}{\partial t} \\ \frac{\partial \alpha_q}{\partial t} \end{pmatrix}, \quad \begin{pmatrix} e_p \\ e_q \end{pmatrix} = \begin{pmatrix} \frac{\delta H}{\delta \alpha_p} \\ \frac{\delta H}{\delta \alpha_q} \end{pmatrix}. \tag{71}$$

Their product equals again the time-variation of the Hamiltonian

$$\frac{dH}{dt} = \int_Z (e_p \wedge f_p + e_q \wedge f_q). \tag{72}$$

Using the conservation laws (4.5) for $g_i = 0$, the closure relations (70) and the properties of the exterior derivative d and Stokes' theorem, one obtains

$$\begin{aligned} \frac{dH}{dt} &= \int_Z \varepsilon \beta_q \wedge (-d\beta_p) + (-1)^r \beta_p \wedge \varepsilon(-d\beta_q) \\ &= -\varepsilon \int_Z \beta_q \wedge d\beta_p + (-1)^q \beta_q \wedge d\beta_p = -\varepsilon \int_{\partial Z} \beta_q \wedge \beta_p. \end{aligned} \tag{73}$$

Finally, as before we define the power-conjugated pair of *flow and effort variables on the boundary* as the *restriction* of the flux variables to the boundary ∂Z of the domain Z:

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \begin{pmatrix} \beta_q|_{\partial Z} \\ \beta_p|_{\partial Z} \end{pmatrix}. \tag{74}$$

On the total space of power-conjugated variables, that is, the differential forms (f_p, e_p) and (f_q, e_q) on the domain Z and the differential forms (f_∂, e_∂) defined on the boundary ∂Z, one defines an *interconnection structure* by Eqn. (74) together with

$$\begin{pmatrix} f_q \\ f_p \end{pmatrix} = \varepsilon \begin{pmatrix} 0 & (-1)^r d \\ d & 0 \end{pmatrix} \begin{pmatrix} e_q \\ e_p \end{pmatrix}. \tag{75}$$

This interconnection can be formalized as a special type of Dirac structure, called Stokes–Dirac structure, leading to the definition of distributed-parameter port-Hamiltonian systems [35].

5. Concluding remarks

We have surveyed some of the recently developed theory of port-Hamiltonian systems; for further applications towards modeling, analysis, simulation and control we refer to the literature cited below.

From the geometric point of view many questions regarding port-Hamiltonian systems are waiting to be investigated. A theory of symmetry and reduction of port-Hamiltonian systems has been explored in [29], [1], while some questions concerning integrability of Dirac structures have been studied in [7]. A main question for distributed-parameter port-Hamiltonian systems concerns the relation with variational calculus.

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