Passive linear discrete time-invariant systems

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Abstract. We begin by discussing linear discrete time-invariant i/s/o (input/state/output) systems that satisfy certain ‘energy’ inequalities. These inequalities involve a quadratic storage function in the state space induced by a positive self-adjoint operator H that may be unbounded and have an unbounded inverse, and also a quadratic supply rate in the combined i/o (input/output) space. The three most commonly studied classes of supply rates are called scattering, impedance, and transmission. Although these three classes resemble each other, we show that there are still significant differences. We then present a new class of s/s (state/signal) systems which have a Hilbert state space and a Kreĭn signal space. The state space is used to store relevant information about the past evolution of the system, and the signal space is used to describe interactions with the surrounding world. A s/s system resembles an i/s/o system apart from the fact that inputs and outputs are not separated from each other. By decomposing the signal space into a direct sum of an input space and an output space one gets a standard i/s/o system, provided the decomposition is *admissible*, and different i/o decompositions lead to different i/o supply rates (for example of scattering, impedance, or transmission type). In the case of non-admissible decompositions we obtain right and left affine representations, both of the s/s system itself, and of the corresponding transfer function. In particular, in the case of a passive system we obtain right and left coprime representations of the generalized transfer functions corresponding to nonadmissible decompositions of the signal space, and we end up with transfer functions which are, e.g., generalized Potapov or Nevanlinna class functions.

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1. H-passive discrete time i/s/o systems

The evolution of a *linear discrete time-invariant i/s/o* (*input/state/output*) *system* $\Sigma_{\mathrm{i/s/o}}$ with a Hilbert *input space* $\mathcal{U}$, a Hilbert *state space* $\mathcal{X}$, and a Hilbert *output space* $\mathcal{Y}$ is described by the system of equations

$$\begin{aligned} x(n+1) &= Ax(n) + Bu(n), \\ y(n) &= Cx(n) + Du(n), \quad n \in \mathbb{Z}^+ = \{0, 1, 2, \dots\}, \\ x(0) &= x_0, \end{aligned} \tag{1.1}$$

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where the initial state $x_0 \in \mathcal{X}$ may be chosen arbitrarily and $A\colon \mathcal{X} \to \mathcal{X}$, $B\colon \mathcal{U} \to \mathcal{X}$, $C\colon \mathcal{X} \to \mathcal{Y}$, and $D\colon \mathcal{U} \to \mathcal{Y}$ are bounded linear operators. Equivalently,

$$\begin{bmatrix} x(n+1) \\ y(n) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x(n) \\ u(n) \end{bmatrix}, \quad n \in \mathbb{Z}^+,\ x(0) = x_0, \tag{1.2}$$

where $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{B}\left(\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]; \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]\right)$.[1] We call $u = \{u(n)\}_{n=0}^{\infty}$ the *input sequence*, $x = \{x(n)\}_{n=0}^{\infty}$ the *state trajectory*, and $y = \{y(n)\}_{n=0}^{\infty}$ the *output sequence*, and we refer to the triple (u, x, y) as a *trajectory of* $\Sigma_{\mathrm{i/s/o}}$. The operators appearing in (1.1) and (1.2) are usually called as follows: A is the *main operator*, B is the *control operator*, C is the *observation operator*, and D is the *feedthrough operator*. The *transfer function* or *characteristic function* $\mathfrak{D}$ of this system is given by[2]

$$\mathfrak{D}(z) = zC(1_{\mathcal{X}} - zA)^{-1}B + D, \quad z \in \Lambda(A),$$

where $\Lambda(A)$ is the set of points $z \in \mathbb{C}$ for which $1_{\mathcal{X}} - zA$ has a bounded inverse, plus the point at infinity if A has a bounded inverse. Note that $\mathfrak{D}$ is analytic on $\Lambda(A)$, and that $D = \mathfrak{D}(0)$. We shall denote the above system by $\Sigma_{\mathrm{i/s/o}} = \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{U}, \mathcal{X}, \mathcal{Y}\right)$. Since all the systems in this paper will be linear and time-invariant and have a discrete time variable we shall in the sequel omit the words “linear discrete time-invariant” and refer to a system of the above type by simply calling it an *i/s/o system*.

The i/s/o system $\Sigma_{\mathrm{i/s/o}}$ is *controllable* if the sets of all states $x(n)$, $n \geq 1$, which appear in some trajectory (u, x, y) of $\Sigma_{\mathrm{i/s/o}}$ with $x_0 = 0$ (i.e., an *externally generated trajectory*) is dense in $\mathcal{X}$. The system $\Sigma_{\mathrm{i/s/o}}$ is *observable* if there do not exist any nontrivial trajectories (u, x, y) where both u and y are identically zero. Finally, $\Sigma_{\mathrm{i/s/o}}$ is *minimal* if $\Sigma_{\mathrm{i/s/o}}$ is both controllable and observable.

In this work we shall primarily be concerned with i/s/o systems which are passive or even conservative. To define these notions we first introduce the notions of a storage function E_H which represents the (internal) energy of the state, and a supply rate j which describes the interchange of energy between the system and its surroundings. In the classical case the *storage* (or *Lyapunov*) *function* E_H is bounded, and it is given by $E_H(x) = \langle x, Hx \rangle_{\mathcal{X}}$, where H is a bounded positive self-adjoint operator on $\mathcal{X}$ (positivity of H means that $\langle x, Hx \rangle_{\mathcal{X}} > 0$ for all $x \neq 0$). However, we shall also consider unbounded storage functions induced by some (possibly unbounded) positive self-adjoint operator H on $\mathcal{X}$. In this case we let $\sqrt{H}$ be the positive self-adjoint square root of H, and define the storage function E_H by

$$E_H(x) = \|\sqrt{H}x\|_{\mathcal{X}}^2, \quad x \in \mathcal{D}(\sqrt{H}). \tag{1.3}$$

Clearly, this is equivalent to the earlier definition of E_H if H is bounded. The *supply rate* j will always be a bounded (indefinite) self-adjoint quadratic form on $\mathcal{Y} \oplus \mathcal{U}$,

[1] Here $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$ is the cartesian product of $\mathcal{X}$ and $\mathcal{U}$, and $\mathcal{B}(\mathcal{U}; \mathcal{Y})$ is the set of bounded linear operators from $\mathcal{U}$ to $\mathcal{Y}$.

[2] $1_{\mathcal{X}}$ is the identity operator in $\mathcal{X}$.

i.e., it can be written in the form

$$j(u, y) = \left\langle \begin{bmatrix} y \\ u \end{bmatrix}, J \begin{bmatrix} y \\ u \end{bmatrix} \right\rangle_{\mathcal{Y}\oplus\mathcal{U}}, \tag{1.4}$$

where $J = \left[\begin{smallmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{smallmatrix}\right]$ is a bounded self-adjoint operator in $\mathcal{Y} \oplus \mathcal{U}$. For simplicity we throughout require J to have a bounded inverse. Often J is taken to be a signature operator (both self-adjoint and unitary), so that $J = J^* = J^{-1}$. In the sequel we shall always use *one and the same supply rate* j for a given system $\Sigma_{\mathrm{i/s/o}}$ and include this supply rate in the notation of the system, thus denoting the system by $\Sigma_{\mathrm{i/s/o}} = \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{U}, \mathcal{X}, \mathcal{Y}; j\right)$ whenever the supply rate is important.

Definition 1.1. The i/s/o system $\Sigma_{\mathrm{i/s/o}} = \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{U}, \mathcal{X}, \mathcal{Y}; j\right)$ is *forward H-passive*, where H is a positive self-adjoint operator in $\mathcal{X}$, if $x(n) \in \mathcal{D}(\sqrt{H})$ and

$$\|\sqrt{H}x(n+1)\|^2_{\mathcal{X}} - \|\sqrt{H}x(n)\|^2_{\mathcal{X}} \leq j(u(n), y(n)), \quad n \in \mathbb{Z}^+, \tag{1.5}$$

for every trajectory (u, x, y) of $\Sigma_{\mathrm{i/s/o}}$ with $x_0 \in \mathcal{D}(\sqrt{H})$. If the above inequality holds as an equality then $\Sigma_{\mathrm{i/s/o}}$ is *forward H-conservative*.

It is not difficult to see that $\Sigma_{\mathrm{i/s/o}}$ is forward H-passive if and only if[3] $H > 0$ is a solution of the (forward) generalized i/s/o *KYP* (*Kalman–Yakubovich–Popov*) inequality[4]

$$\|\sqrt{H}(Ax + Bu)\|^2_{\mathcal{X}} - \|\sqrt{H}x\|^2_{\mathcal{X}} \leq j(u, Cx + Du), \quad x \in \mathcal{D}(\sqrt{H}),\ u \in \mathcal{U}, \tag{1.6}$$

and that $\Sigma_{\mathrm{i/s/o}}$ is forward H-conservative if and only if $H > 0$ is a solution of the corresponding equality. This inequality is named after *Kalman* [Kal63], *Yakubovich* [Yak62], and *Popov* [Pop61] (who at that time restricted themselves to the finite-dimensional case). There is a rich literature on the finite-dimensional version of the KYP inequality and the corresponding equality; see, e.g., [PAJ91], [IW93] and [LR95], and the references mentioned there. In the seventies the classical results on the KYP inequalities were extended to infinite-dimensional systems by V. A. Yakubovich and his students and collaborators (see [Yak74], [Yak75], and [LY76] and the references listed there). There is now also a rich literature on this infinite-dimensional case; see, e.g., the discussion in [Pan99] and the references cited there. However, until recently it was assumed throughout that *either H itself is bounded or H^{-1} is bounded*. The first study of this inequality which permits both H and H^{-1} to be unbounded was done by Arov, Kaashoek and Pik in [AKP05].

Above we have defined *forward H*-passivity and forward H-conservativity. The corresponding *backward* notions are defined by means of the adjoint i/s/o system

[3] The notation $H > 0$ means that H is a (possibly unbounded) self-adjoint operator satisfying $\langle x, Hx\rangle_{\mathcal{X}} > 0$ for all nonzero $x \in \mathcal{D}(H)$.

[4] In particular, in order for the first term in this inequality to be well-defined we require A to map $\mathcal{D}(\sqrt{H})$ into itself and B to map $\mathcal{U}$ into $\mathcal{D}(\sqrt{H})$.

$\Sigma^*_{\mathrm{i/s/o}} = \left(\left[\begin{smallmatrix} A^* & C^* \\ B^* & D^* \end{smallmatrix}\right]; \mathcal{Y}, \mathcal{X}, \mathcal{U}; j_*\right)$ whose trajectories (y_*, x_*, u_*) satisfy the system of equations

$$\begin{aligned} x_*(n+1) &= A^* x_*(n) + C^* y_*(n), \\ u_*(n) &= B^* x_*(n) + D^* y_*(n), \quad n \in \mathbb{Z}^+, \\ x_*(0) &= x_{*0}. \end{aligned} \tag{1.7}$$

Note that this system has the same state space $\mathcal{X}$, but the input and output have been interchanged, so that $\mathcal{Y}$ is the input space and $\mathcal{U}$ is the output space. The appropriate storage function and supply rates for the adjoint system $\Sigma^*_{\mathrm{i/s/o}}$ differ from those of the primal system $\Sigma_{\mathrm{i/s/o}}$: H is replaced by H^{-1}, and the primal supply rate j is replaced by the dual supply rate

$$j_*(y_*, u_*) = \left\langle \begin{bmatrix} u_* \\ y_* \end{bmatrix}, J_* \begin{bmatrix} u_* \\ y_* \end{bmatrix} \right\rangle_{\mathcal{U}\oplus\mathcal{Y}}, \tag{1.8}$$

where

$$J_* = \begin{bmatrix} 0 & -1_{\mathcal{U}} \\ 1_{\mathcal{Y}} & 0 \end{bmatrix} J^{-1} \begin{bmatrix} 0 & -1_{\mathcal{Y}} \\ 1_{\mathcal{U}} & 0 \end{bmatrix}. \tag{1.9}$$

Definition 1.2. Let $\Sigma_{\mathrm{i/s/o}} = \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{U}, \mathcal{X}, \mathcal{Y}; j\right)$ be an i/s/o system, and let H be a positive self-adjoint operator in $\mathcal{X}$.

(i) $\Sigma_{\mathrm{i/s/o}}$ is *backward H-passive* if the adjoint system $\Sigma^*_{\mathrm{i/s/o}}$ is forward H^{-1}-passive.

(ii) $\Sigma_{\mathrm{i/s/o}}$ is *backward H-conservative* if the adjoint system $\Sigma^*_{\mathrm{i/s/o}}$ is forward H^{-1}-conservative.

(iii) $\Sigma_{\mathrm{i/s/o}}$ is H-passive if it is both forward and backward H-passive.

(iv) $\Sigma_{\mathrm{i/s/o}}$ is H-conservative if it is both forward and backward H-conservative.

(v) By *passive* or *conservative* (with or without the attributes "forward" or "backward") we mean $1_{\mathcal{X}}$-passive or $1_{\mathcal{X}}$-conservative, respectively.

The generalized KYP inequality for the adjoint i/s/o system $\Sigma^*_{\mathrm{i/s/o}}$ with storage function $E_{H^{-1}}$ is given by[5]

$$\begin{aligned} \|H^{-1/2}(A^* x_* + C^* y_*)\|^2_{\mathcal{X}} - \|H^{-1/2} x_*\|^2_{\mathcal{X}} \le j_*(y_*, B^* x_* + D^* y_*), \\ x_* \in \left(\sqrt{H}\right), \ y_* \in \mathcal{Y}. \end{aligned} \tag{1.10}$$

Thus, $\Sigma_{\mathrm{i/s/o}}$ is backward H-passive if and only if H is a solution of (1.10), and $\Sigma_{\mathrm{i/s/o}}$ is backward H-conservative if and only if H is a solution of the corresponding equality.

[5] In particular, in order for the first term in this inequality to be well-defined we require A^* to map $\mathcal{R}(\sqrt{H})$ into itself and C^* to map $\mathcal{Y}$ into $\mathcal{R}(\sqrt{H})$.

2. Scattering, impedance and transmission supply rates

The three most common supply rates are the following:

(i) The *scattering* supply rate $j_{\mathrm{sca}}(u, y) = -\langle y, y\rangle_{\mathcal{Y}} + \langle u, u\rangle_{\mathcal{U}}$ with signature operator $J_{\mathrm{sca}} = \begin{bmatrix} -1_{\mathcal{Y}} & 0 \\ 0 & 1_{\mathcal{U}} \end{bmatrix}$. The signature operator of the dual supply rate is $J_{\mathrm{sca}*} = \begin{bmatrix} -1_{\mathcal{U}} & 0 \\ 0 & 1_{\mathcal{Y}} \end{bmatrix}$.

(ii) The *impedance* supply rate $j_{\mathrm{imp}}(u, y) = 2\Re\langle y, \Psi u\rangle_{\mathcal{U}}$ with signature operator $J_{\mathrm{imp}} = \begin{bmatrix} 0 & \Psi \\ \Psi^* & 0 \end{bmatrix}$, where Ψ is a unitary operator $\mathcal{U} \to \mathcal{Y}$. The signature operator of the dual supply rate is $J_{\mathrm{imp}*} = \begin{bmatrix} 0 & \Psi^* \\ \Psi & 0 \end{bmatrix}$.

(iii) The *transmission* supply rate $j_{\mathrm{tra}}(u, y) = -\langle y, J_{\mathcal{Y}} y\rangle_{\mathcal{Y}} + \langle u, J_{\mathcal{U}} u\rangle_{\mathcal{U}}$ with signature operator $J_{\mathrm{tra}} = \begin{bmatrix} -J_{\mathcal{Y}} & 0 \\ 0 & J_{\mathcal{U}} \end{bmatrix}$, where $J_{\mathcal{Y}}$ and $J_{\mathcal{U}}$ are signature operators in $\mathcal{Y}$ and $\mathcal{U}$, respectively. The signature operator of the dual supply rate is $J_{\mathrm{tra}*} = \begin{bmatrix} -J_{\mathcal{U}} & 0 \\ 0 & J_{\mathcal{Y}} \end{bmatrix}$.

In the sequel when we talk about *scattering H-passive* or *impedance H-conservative*, etc., we mean that the supply rate is of the corresponding type. It turns out that although Definition 1.1 and 1.2 can be applied to all three types of supply rates, these three cases still differ significantly from each other.

2.1. Scattering supply rate. In the case of scattering supply rate *forward H-passivity is equivalent to backward H-passivity, hence to passivity*. This is easy to see in the case where $H = 1_{\mathcal{X}}$: the system $\Sigma_{\mathrm{i/s/o}} = \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}; \mathcal{U}, \mathcal{X}, \mathcal{Y}; j_{\mathrm{sca}}\right)$ is forward passive if and only if the operator $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is a contraction, which is true if and only if its adjoint $\begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix}$ is a contraction, which is true if and only if the adjoint system $\Sigma^*_{\mathrm{i/s/o}} = \left(\begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix}; \mathcal{U}, \mathcal{X}, \mathcal{Y}; j_{\mathrm{sca}*}\right)$ is forward passive. The case where H is bounded and has a bounded inverse is almost as easy, and the general case is proved in [AKP05, Proposition 4.6].

The existence of an operator $H > 0$ such that $\Sigma_{\mathrm{i/s/o}}$ is scattering H-passive is related to the properties of the transfer function $\Sigma_{\mathrm{i/s/o}}$. To formulate this result we first recall some definitions. The *Schur class* $\mathcal{S}(\mathcal{U}, \mathcal{Y}; \mathbb{D})$ is the unit ball in $H^\infty(\mathcal{U}, \mathcal{Y}, \mathbb{D})$, i.e., each function in $\mathcal{S}(\mathcal{U}, \mathcal{Y}; \mathbb{D})$ is an analytic function on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ whose values are contractions in $\mathcal{B}(\mathcal{U}, \mathcal{Y})$. The *restricted Schur class* $\mathcal{S}(\mathcal{U}, \mathcal{Y}; \Omega)$, where $\Omega \subset \mathbb{D}$, contains all functions θ which are restrictions to Ω of some function in $\mathcal{S}(\mathcal{U}, \mathcal{Y}; \mathbb{D})$. In other words, $\theta \in \mathcal{S}(\mathcal{U}, \mathcal{Y}; \Omega)$ if the (Nevanlinna–Pick) extension (or interpolation) problem with the (possibly infinite) set of data points $(z, \theta(z))$, $z \in \Omega$, has a solution in $\mathcal{S}(\mathcal{U}, \mathcal{Y}; \mathbb{D})$. It is known that this problem has a solution if and only if the kernel

$$K^\theta_{\mathrm{sca}}(z, \zeta) = \frac{1_{\mathcal{Y}} - \theta(z)\theta(\zeta)^*}{1 - z\overline{\zeta}}, \quad z, \zeta \in \Omega,$$

is nonnegative definite on $\Omega \times \Omega$, or equivalently, if and only if the kernel

$$K_{\mathrm{sca}}^{\theta *}(z,\zeta) = \frac{1_{\mathcal{U}} - \theta(\zeta)^*\theta(z)}{1-\overline{\zeta}z}, \quad z,\zeta \in \Omega,$$

is nonnegative definite on $\Omega \times \Omega$ (see [RR82]). We shall here be interested in the case where Ω is an *open* subset of $\mathbb{D}$, which implies that the solution of this Nevanlinna–Pick extension problem is unique (if it exists).

Theorem 2.1. *Let $\Sigma_{\mathrm{i/s/o}} = \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{U}, \mathcal{X}, \mathcal{Y}; j_{\mathrm{sca}}\right)$ be an i/s/o system with scattering supply rate and transfer function $\mathfrak{D}$, and let $\Lambda_0(A)$ be the connected component of $\Lambda(A) \cap \mathbb{D}$ which contains the origin.*

(i) *If $\Sigma_{\mathrm{i/s/o}}$ is forward H-passive for some $H > 0$, then $\Sigma_{\mathrm{i/s/o}}$ is H-passive and $\mathfrak{D}|_{\Lambda_0(A)} \in \mathcal{S}(\mathcal{U}, \mathcal{Y}; \Lambda_0(A))$.*

(ii) *Conversely, if $\Sigma_{\mathrm{i/s/o}}$ is minimal and $\mathfrak{D}|_{\Lambda_0(A)} \in \mathcal{S}(\mathcal{U}, \mathcal{Y}; \Lambda_0(A))$, then $\Sigma_{\mathrm{i/s/o}}$ is H-passive for some $H > 0$.*

In statement (ii) it is actually possible to choose the operator H to satisfy an additional minimality requirement. We shall return to this question in Theorem 3.5.

2.2. Impedance supply rate. Also in the case of impedance supply rate *forward H-passivity is equivalent to backward H-passivity, hence to passivity*. This is well known in the case where $H = 1_{\mathcal{X}}$ (see, e.g., [Aro79a]). One way to prove this is to reduce the impedance case to the scattering case by means of the following simple transformation.

Suppose that $\Sigma_{\mathrm{i/s/o}} = \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{U}, \mathcal{X}, \mathcal{Y}; j_{\mathrm{imp}}\right)$ is a forward impedance H-passive system with signature operator $J_{\mathrm{imp}} = \left[\begin{smallmatrix} 0 & \Psi \\ \Psi^* & 0 \end{smallmatrix}\right]$. Let (u, x, y) be a trajectory of $\Sigma_{\mathrm{i/s/o}}$. We define a new input $u^\times$ by $u^\times = \frac{1}{\sqrt{2}}(u + \Psi^* y)$ and a new output $y^\times$ by $y^\times = \frac{1}{\sqrt{2}}(\Psi u - y)$, after which we solve (1.2) for x and $y^\times$ in terms of x_0 and $u^\times$. It turns out that for this to be possible we need $\Psi + D$ to have a bounded inverse. However, this is always the case, since (1.6) (with $x = 0$) implies that $\Psi^* D + D^* \Psi \geq 0$. A direct computation shows that $(y^\times, x, u^\times)$ is a trajectory of another system $\Sigma_{\mathrm{i/s/o}}^\times = \left(\left[\begin{smallmatrix} A^\times & B^\times \\ C^\times & D^\times \end{smallmatrix}\right]; \mathcal{U}, \mathcal{X}, \mathcal{Y}\right)$, called the *external Cayley transform of* $\Sigma_{\mathrm{i/s/o}}$, whose coefficients are given by

$$\begin{aligned} A^\times &= A - B(\Psi + D)^{-1}C, & B^\times &= \sqrt{2}\, B(\Psi + D)^{-1}\Psi, \\ C^\times &= -\sqrt{2}\, \Psi(\Psi + D)^{-1}C, & D^\times &= (\Psi - D)(\Psi + D)^{-1}\Psi. \end{aligned} \tag{2.1}$$

The transfer functions of the two systems are connected by

$$\mathfrak{D}^\times(z) = (\Psi - \mathfrak{D}(z))(\Psi + \mathfrak{D}(z))^{-1}\Psi, \quad z \in \Lambda(A) \cap \Lambda(A^\times). \tag{2.2}$$

The external Cayley transform is its own inverse in the sense that $\Psi + D^{\times} = 2\Psi(\Psi + D)^{-1}\Psi$ always has a bounded inverse, and if we apply the external Cayley transform to the system $\Sigma^{\times}_{\mathrm{i/s/o}}$, then we recover the original system $\Sigma_{\mathrm{i/s/o}}$.

The main reason for defining the external Cayley transform in the way that we did above is that it 'preserves the energy exchange' in the sense that $j_{\mathrm{imp}}(u, y) = j_{\mathrm{sca}}(y^{\times}, u^{\times})$. This immediately implies that $\Sigma^{\times}_{\mathrm{i/s/o}}$ is forward scattering H-passive whenever $\Sigma_{\mathrm{i/s/o}}$ is forward impedance H-passive.[6] According to the discussion in Section 2.1, forward scattering H-passivity of $\Sigma^{\times}_{\mathrm{i/s/o}}$ is equivalent to backward scattering H-passivity of $\Sigma^{\times}_{\mathrm{i/s/o}}$, and this in turn is equivalent to the backward (impedance) H-passivity of $\Sigma_{\mathrm{i/s/o}}$. Thus, we get the desired conclusion, namely that forward impedance H-passivity implies backward impedance H-passivity, hence impedance H-passivity.

The same argument can be used to convert all the results mentioned in Section 2.1 into an impedance setting. For simplicity we below take $\mathcal{Y} = \mathcal{U}$ and $\Psi = 1_{\mathcal{U}}$ (this amounts to replacing the output sequence y with values in $\mathcal{Y}$ by the new output sequence $\Psi^* y$ with values in $\mathcal{U}$)). The *Carathéodory class* $\mathcal{C}(\mathcal{U}; \mathbb{D})$ (also called the Carathéodory–Nevanlinna class, or Nevanlinna class, or Weyl class, or Titchmarsh–Weyl class, etc.) consists of all analytic $\mathcal{B}(\mathcal{U})$-valued functions ψ on $\mathbb{D}$ with nonnegative 'real part', i.e., $\psi(z) + \psi(z)^* \geq 0$ for all $z \in \mathbb{D}$. The *restricted Carathéodory class* $\mathcal{C}(\mathcal{U}; \Omega)$, where $\Omega \subset \mathbb{D}$, contains all functions θ which are restrictions to Ω of some function in $\mathcal{C}(\mathcal{U}; \mathbb{D})$. In other words, $\theta \in \mathcal{C}(\mathcal{U}; \Omega)$ if the extension problem with the set of data points $(z, \theta(z))$, $z \in \Omega$, has a solution in $\mathcal{C}(\mathcal{U}; \Omega)$. This is equivalent to the requirement that the kernel

$$K^{\psi}_{\mathrm{imp}}(z, \zeta) = \frac{\psi(z) + \psi(\zeta)^*}{1 - z\bar{\zeta}}, \quad z, \zeta \in \Omega,$$

is nonnegative definite on $\Omega \times \Omega$ (this can be proved by reducing the impedance case to the scattering case as explained above).

Theorem 2.2. *Let* $\Sigma_{\mathrm{i/s/o}} = \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{U}, \mathcal{X}, \mathcal{U}; j_{\mathrm{imp}}\right)$ *be an i/s/o system with impedance supply rate, signature operator* $J_{\mathrm{imp}} = \begin{bmatrix} 0 & 1_{\mathcal{U}} \\ 1_{\mathcal{U}} & 0 \end{bmatrix}$*, and transfer function* $\mathfrak{D}$*. Let* $\Lambda_0(A)$ *be the connected component of* $\Lambda(A) \cap \mathbb{D}$ *which contains the origin.*

(i) *If* $\Sigma_{\mathrm{i/s/o}}$ *is forward H-passive for some* $H > 0$*, then* $\Sigma_{\mathrm{i/s/o}}$ *is H-passive and* $\mathfrak{D}|_{\Lambda_0(A)} \in \mathcal{C}(\mathcal{U}, \mathcal{Y}; \Lambda_0(A))$.

(ii) *Conversely, if* $\Sigma_{\mathrm{i/s/o}}$ *is minimal and* $\mathfrak{D}|_{\Lambda_0(A)} \in \mathcal{C}(\mathcal{U}, \mathcal{Y}; \Lambda_0(A))$*, then* $\Sigma_{\mathrm{i/s/o}}$ *is H-passive for some* $H > 0$.

This theorem follows from Theorem 2.1 as explained above.

[6] It is also true that $\Sigma^{\times}_{\mathrm{i/s/o}}$ is forward impedance H-passive if $\Sigma_{\mathrm{i/s/o}}$ is forward scattering H-passive, provided $(\Psi + D)$ has a bounded inverse so that $\Sigma^{\times}_{\mathrm{i/s/o}}$ exists.

Above we have reduced the impedance passive case to the scattering passive case. Historically the development went in the opposite direction: the impedance version is older than the scattering version. It is related to Neumark's dilation theorem for positive operator-valued measures (see [Bro71, Appendix 1]). In many classical and also in some recent works (especially those where the functions are defined on a half-plane instead of the unit disk) the impedance version is used as 'reference system' from which scattering and other results are derived (see, e.g., [Bro78]). Thus, one easily arrives at the (in my opinion incorrect) conclusion that it does not really matter which one of the two classes is used as the basic corner stone on which the theory is built. However, there is a significant difference between the two classes: the *external Cayley transformation* that converts one of the classes into the other *is well-defined for every impedance H-passive system, but not for every scattering H-passive system*. In other words, the external Cayley transform maps the class of impedance H-passive systems *into but not onto* the class of scattering H-passive systems (even if we restrict the input and output dimensions of the scattering system to be the same).

What happens if we try to apply the external Cayley transform to a scattering H-passive system for which this transform is not defined (i.e., $\Psi + D$ is not invertible)? In this case the formal transfer function of the resulting system may take its values in the space of closed unbounded operators in $\mathcal{U}$, and it may even be multi-valued. To be able to study this class of 'generalized Carathéodory functions' we need some other more general type of linear systems than the i/s/o systems we have considered so far. This was one of the motivations for the introduction of the notion of a state/signal system in [AS05], to be discussed in Section 3.

2.3. Transmission supply rate. In the case of transmission supply rate *forward H-passivity is no longer equivalent to backward H-passivity*. For simplicity, let us take H to be the identity. Arguing in the same way as in the scattering case we find that $\Sigma_{\mathrm{i/s/o}} = \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{U}, \mathcal{X}, \mathcal{Y}; j_{\mathrm{tra}}\right)$ is forward (transmission) passive if and only if the operator $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is a contraction[7] between two Kreĭn spaces, namely from the space $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$ with the signature operator $\begin{bmatrix} 1_{\mathcal{X}} & 0 \\ 0 & J_{\mathcal{U}} \end{bmatrix}$ to the space $\begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$ with the signature operator $\begin{bmatrix} 1_{\mathcal{X}} & 0 \\ 0 & J_{\mathcal{Y}} \end{bmatrix}$. In the same way we find that $\Sigma_{\mathrm{i/s/o}}$ is backward (transmission) passive if $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]^*$ is a contraction from the space $\begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$ with the signature operator $\begin{bmatrix} 1_{\mathcal{X}} & 0 \\ 0 & J_{\mathcal{Y}} \end{bmatrix}$ to the space $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$ with the signature operator $\begin{bmatrix} 1_{\mathcal{X}} & 0 \\ 0 & J_{\mathcal{U}} \end{bmatrix}$. However, *in a Kreĭn space setting the contractivity of an operator does not imply that the adjoint of this operator is contractive*, and hence forward transmission passivity does not imply backward transmission passivity without any further restrictions on the system. One necessary condition for the system $\Sigma_{\mathrm{i/s/o}}$ to be both forward and backward (transmission) H-passive is that the *dimensions of the negative eigenspaces of $J_{\mathcal{U}}$ and $J_{\mathcal{Y}}$ are the*

[7] An operator $A \in \mathcal{B}(\mathcal{U}; \mathcal{Y})$, where $\mathcal{U}$ and $\mathcal{Y}$ are Kreĭn spaces, is a contraction if $[Au, Au]_{\mathcal{Y}} \le [u, u]_{\mathcal{U}}$ for all $u \in \mathcal{U}$.

same. If these dimensions are the same *and finite*, then it is true that forward H-passivity is equivalent to backward H-passivity, hence to passivity. To prove these statements one can use the following transformation that maps the transmission supply rate into a scattering supply rate.

Suppose that $\Sigma_{\mathrm{i/s/o}} = \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{U}, \mathcal{X}, \mathcal{Y}; j_{\mathrm{tra}}\right)$ is a forward transmission H-passive system with signature operator $J_{\mathrm{tra}} = \left[\begin{smallmatrix} J_{\mathcal{Y}} & 0 \\ 0 & J_{\mathcal{Y}} \end{smallmatrix}\right]$. We begin by splitting the output space $\mathcal{Y}$ into the orthogonal direct sum $\mathcal{Y} = -\mathcal{Y}_- \,[\dot{+}]\, \mathcal{Y}_+$, where $\mathcal{Y}_-$ is the negative and $\mathcal{Y}_+$ is the positive eigenspace of $J_{\mathcal{Y}}$. In the same way we split the input space $\mathcal{U}$ into $\mathcal{U} = -\mathcal{U}_- \,[\dot{+}]\, \mathcal{U}_+$, and we split the operator $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ accordingly into

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|cc} A & B_1 & B_1 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right].$$

Let (u, x, y) be a trajectory of $\Sigma_{\mathrm{i/s/o}}$, and split y and u into $y = \left[\begin{smallmatrix} y_- \\ y_+ \end{smallmatrix}\right]$ and $u = \left[\begin{smallmatrix} u_- \\ u_+ \end{smallmatrix}\right]$, so that y_- is a sequence in $\mathcal{Y}_-$, etc. We define a new input $u^{\curvearrowleft}$ by $u^{\curvearrowleft} = \left[\begin{smallmatrix} y_- \\ u_+ \end{smallmatrix}\right]$ and a new output $y^{\curvearrowleft}$ by $y^{\curvearrowleft} = \left[\begin{smallmatrix} u_- \\ y_+ \end{smallmatrix}\right]$, so that $u^{\curvearrowleft}$ is a sequence in $\mathcal{U}^{\curvearrowleft} = \mathcal{Y}_- \oplus \mathcal{U}_+$ and $y^{\curvearrowleft}$ is a sequence in $\mathcal{Y}^{\curvearrowleft} = \mathcal{U}_- \oplus \mathcal{Y}_+$. We then solve (1.2) for x and $y^{\curvearrowleft}$ in terms of x_0 and $u^{\curvearrowleft}$. It turns out that for this to be possible we need D_{11} to have a bounded inverse. The forward H-passivity of $\Sigma_{\mathrm{i/s/o}}$ implies that D_{11} is injective and has a closed range, but it need not be surjective. However, let us suppose that D_{11} has a bounded inverse. Then a direct computation shows that $(u^{\curvearrowleft}, x, y^{\curvearrowleft})$ is a trajectory of another system $\Sigma^{\curvearrowleft}_{\mathrm{i/s/o}} = \left(\left[\begin{smallmatrix} A^{\curvearrowleft} & B^{\curvearrowleft} \\ C^{\curvearrowleft} & D^{\curvearrowleft} \end{smallmatrix}\right]; \mathcal{U}^{\curvearrowleft}, \mathcal{X}, \mathcal{Y}^{\curvearrowleft}\right)$, called the *Potapov–Ginzburg* (or *chain scattering*) *transform of* $\Sigma_{\mathrm{i/s/o}}$, whose coefficients are given by

$$\begin{aligned} \begin{bmatrix} A^{\curvearrowleft} & B^{\curvearrowleft} \\ C^{\curvearrowleft} & D^{\curvearrowleft} \end{bmatrix} &= \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline 0 & 1_{\mathcal{Y}_-} & 0 \\ C_2 & D_{21} & D_{22} \end{array}\right] \left[\begin{array}{c|cc} 1_{\mathcal{X}} & 0 & 0 \\ \hline C_1 & D_{11} & D_{12} \\ 0 & 0 & 1_{\mathcal{U}_+} \end{array}\right]^{-1} \\ &= \left[\begin{array}{c|cc} 1_{\mathcal{X}} & -B_1 & 0 \\ \hline 0 & -D_{11} & 0 \\ 0 & -D_{21} & 1_{\mathcal{Y}_+} \end{array}\right]^{-1} \left[\begin{array}{c|cc} A & 0 & B_2 \\ \hline C_1 & -1_{\mathcal{U}_-} & D_{12} \\ C_2 & 0 & D_{22} \end{array}\right]. \end{aligned} \tag{2.3}$$

The transfer functions of the two systems are connected by

$$\begin{aligned} \begin{bmatrix} \mathfrak{D}^{\curvearrowleft}_{11}(z) & \mathfrak{D}^{\curvearrowleft}_{12}(z) \\ \mathfrak{D}^{\curvearrowleft}_{21}(z) & \mathfrak{D}^{\curvearrowleft}_{22}(z) \end{bmatrix} &= \begin{bmatrix} (\mathfrak{D}_{11}(z))^{-1} & -(\mathfrak{D}_{11}(z))^{-1}\mathfrak{D}_{12}(z) \\ \mathfrak{D}_{21}(z)(\mathfrak{D}_{11}(z))^{-1} & \mathfrak{D}_{22}(z) - \mathfrak{D}_{21}(z)(\mathfrak{D}_{11}(z))^{-1}\mathfrak{D}_{12}(z) \end{bmatrix}, \\ & z \in \Lambda(A) \cap \Lambda\left(A^{\curvearrowleft}\right). \end{aligned} \tag{2.4}$$

The Potapov–Ginzburg transform is its own inverse in the sense that $D^{\curvearrowleft}_{11} = D_{11}^{-1}$ always has a bounded inverse, and if we apply the Potapov–Ginzburg transform to the system $\Sigma^{\curvearrowleft}_{\mathrm{i/s/o}}$, then we recover the original system $\Sigma_{\mathrm{i/s/o}}$.

The Potapov–Ginzburg transform has been designed to 'preserve the energy exchange' in the sense that $j_{\rm tra}(u, y) = j_{\rm sca}(u^{\curvearrowleft}, y^{\curvearrowleft})$. This immediately implies that $\Sigma_{\rm i/s/o}^{\curvearrowleft}$ is forward scattering H-passive whenever $\Sigma_{\rm i/s/o}$ is forward transmission H-passive, provided that D_{11} is invertible so that the transform is defined. As in the impedance case we conclude that the forward transmission H-passive system $\Sigma_{\rm i/s/o}$ is also backward H-passive, i.e., H-passive, if D_{11} has a bounded inverse (where D_{11} is the part of the feedthrough operator D that maps the negative part of the input space $\mathcal{U}$ into the negative part of the output space $\mathcal{Y}$). The converse is also true: if $\Sigma_{\rm i/s/o}$ is (transmission) H-passive, then D_{11} has a bounded inverse. Thus, a forward transmission H-passive system $\Sigma_{\rm i/s/o}$ is H-passive if and only if D_{11} has a bounded inverse, or equivalently, if and only if the Potapov–Ginzburg transform of $\Sigma_{\rm i/s/o}$ is defined.

The analogue of Theorems 2.1 and 2.2 is more complicated to formulate than in the scattering and impedance cases. In particular, it is not immediately clear how to define the appropriate class of transfer functions. Above we first defined the Schur class $\mathcal{S}(\mathcal{U}, \mathcal{Y}; \mathbb{D})$ and the Carathéodory class $\mathcal{C}(\mathcal{U}; \mathbb{D})$ in the full unit disk, and then restricted these classes of functions to some subset $\Omega \subset \mathbb{D}$. Here it is easier to proceed in the opposite direction, and to directly define the *restricted Potapov class* $\mathcal{P}(\mathcal{U}, \mathcal{Y}; \Omega)$ for some $\Omega \subset \mathbb{D}$. We now *interpret $\mathcal{U}$ and $\mathcal{Y}$ as Kreĭn spaces*, i.e., we replace the original Hilbert space inner products in $\mathcal{Y}$ and $\mathcal{U}$ by the Kreĭn space inner products

$$[y, y']_{\mathcal{Y}} = \langle y, J_{\mathcal{Y}} y' \rangle_{\mathcal{Y}}, \quad [u, u']_{\mathcal{U}} = \langle u, J_{\mathcal{U}} u' \rangle_{\mathcal{U}}.$$

In the sequel *we compute all adjoints with respect to these Kreĭn space inner products*, and we also *interpret positivity with respect to these inner products* (so that, e.g., an operator D is nonnegative definite in $\mathcal{U}$ if $[u, Du]_{\mathcal{U}} \geq 0$ for all $u \in \mathcal{U}$). A function $\varphi\colon \Omega \to \mathcal{B}(\mathcal{U}; \mathcal{Y})$ belongs to $\mathcal{P}(\mathcal{U}, \mathcal{Y}; \Omega)$ if both the kernels

$$\begin{aligned} K_{\rm tra}^{\varphi}(z, \zeta) &= \frac{1_{\mathcal{Y}} - \varphi(z)\varphi(\zeta)^*}{1 - z\overline{\zeta}}, \quad z, \zeta \in \Omega, \\ K_{\rm tra}^{\varphi *}(z, \zeta) &= \frac{1_{\mathcal{U}} - \varphi^*(\zeta)\varphi(z)}{1 - \overline{\zeta} z}, \quad z, \zeta \in \Omega, \end{aligned} \tag{2.5}$$

are nonnegative definite on $\Omega \times \Omega$.

Theorem 2.3. *Let $\Sigma_{\rm i/s/o} = \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{U}, \mathcal{X}, \mathcal{Y}; j_{\rm tra} \right)$ be an i/s/o system with transmission supply rate, signature operator $J_{\rm tra} = \left[\begin{smallmatrix} J_{\mathcal{Y}} & 0 \\ 0 & J_{\mathcal{U}} \end{smallmatrix}\right]$, and transfer function $\mathfrak{D}$. Let $\Lambda_0(A)$ be the connected component of $\Lambda(A) \cap \mathbb{D}$ which contains the origin.*

(i) *If $\Sigma_{\rm i/s/o}$ is H-passive for some $H > 0$, then $\mathfrak{D}|_{\Lambda_0(A)} \in \mathcal{P}(\mathcal{U}, \mathcal{Y}; \Lambda_0(A))$.*

(ii) *Conversely, if $\Sigma_{\rm i/s/o}$ is minimal and $\mathfrak{D}|_{\Lambda_0(A)} \in \mathcal{P}(\mathcal{U}, \mathcal{Y}; \Lambda_0(A))$, then $\Sigma_{\rm i/s/o}$ is H-passive for some $H > 0$.*

This theorem follows from Theorem 2.1 via the Potapov–Ginzburg transformation. Note that (2.5) with $z = \zeta = 0$ implies that both D and D^* are Kreĭn space contractions, so that D_{11} is invertible and the Potapov–Ginzburg transform is defined.

From what we have said so far it seems to follow that the transmission case is not that different from the scattering and impedances cases. However, this impression is not correct. One significant difference is that the Potapov–Ginzburg transformation is not always defined for a forward transmission H-passive i/s/o system. Another even more serious problem is that a function in the Potapov class may have singularities inside the unit disk $\mathbb{D}$, which means that in the definition of the (full) Potapov class $\mathcal{P}(\mathcal{U}, \mathcal{Y}; \mathbb{D})$ we must take into account that the function in this class need not be defined everywhere on $\mathbb{D}$. If the negative dimensions of $\mathcal{U}$ and $\mathcal{Y}$ are the same and finite, then this is not a serious problem, because in this case it is possible to define the Potapov class $\mathcal{P}(\mathcal{U}, \mathcal{Y}; \mathbb{D})$ to be the set of all meromorphic functions on $\mathbb{D}$ whose values in $\mathcal{B}(\mathcal{U}; \mathcal{Y})$ are contractive with respect to the Kreĭn space inner products in $\mathcal{U}$ and $\mathcal{Y}$ at all points where the functions are defined. However, in the general case the set of singularities of a function in $\mathcal{P}(\mathcal{U}, \mathcal{Y}; \mathbb{D})$ may be uncountable, and the domain of definition of a function in $\mathcal{P}(\mathcal{U}, \mathcal{Y}; \mathbb{D})$ need not even be connected. For this reason we prefer to define $\mathcal{P}(\mathcal{U}, \mathcal{Y}; \mathbb{D})$ in a different way. We say that a function φ belongs to the (full) Potapov class $\mathcal{P}(\mathcal{U}, \mathcal{Y}; \mathbb{D})$ if it belongs to $\mathcal{P}(\mathcal{U}, \mathcal{Y}; \Omega)$ where *the domain* Ω *is maximal* in the sense that the function φ does not have an extension to any larger domain $\Omega' \subset \mathbb{D}$ with the property that the two kernels in (2.5) are still nonnegative on $\Omega' \times \Omega'$. The existence of such a maximal domain is proved in [AS06b]. This maximal domain need not be connected, but it is still true that if we start from an open set $\Omega \subset \mathbb{D}$, then the values of φ on Ω define the extension of φ to its maximal domain uniquely. Moreover, as shown in [AS06b], if $\varphi \in \mathcal{P}(\mathcal{U}, \mathcal{Y}; \mathbb{D})$, then φ does not have an analytic extension to any boundary point of its domain contained in the open unit disk $\mathbb{D}$.

Taking a closer look at Theorem 2.3 we observe that it puts one artificial restriction on the transfer function $\mathfrak{D}$, namely that the domain of definition must contain the origin. Not every function in the Potapov class is defined at the origin, so the class of transfer functions covered by Theorem 2.3 is not the full Potapov class. In addition it is possible to extend the Potapov class so that the values of the functions in this class may be unbounded, even multivalued, operators (as in the impedance case) by taking the formal Potapov transforms of functions in $\mathcal{S}(\mathcal{U}, \mathcal{Y}, \mathbb{D})$. Thus, we again see the need of a more general class of systems than the i/s/o class that we have discussed up to now.

3. State/signal systems

It is possible to develop a linear systems theory where the differences between the three different types of supply rates, namely scattering, impedance, and transmission, more or less disappear. Both the basic transforms that we have presented above, namely the external Cayley transform which is used to pass from an impedance H-passive system to a scattering H-passive system and back, and the Potapov–Ginzburg transform that is used to pass from a transmission H-passive system to a scattering H-passive system and back, can be regarded as simple 'changes of coordinates in the signal space $\mathcal{W} = \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix}$'. The main idea is *not to distinguish between the input sequence u and the output sequence y*, but to simply regard these as components of the general 'signal sequence' $w = \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]$.

We start by combining the input space $\mathcal{U}$ and the output space $\mathcal{Y}$ into one signal space $\mathcal{W} = \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix}$. This signal space has a natural Kreĭn space[8] inner product obtained from the supply rate j in (1.4), namely

$$\left[\!\left[\begin{bmatrix} y \\ u \end{bmatrix}, \begin{bmatrix} y' \\ u' \end{bmatrix}\right]\!\right]_{\mathcal{W}} = \left\langle \begin{bmatrix} y \\ u \end{bmatrix}, J \begin{bmatrix} y' \\ u' \end{bmatrix} \right\rangle_{\mathcal{Y} \oplus \mathcal{U}}.$$

If we combine the input sequence u and the output sequence y into one *signal sequence* $w = \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]$, then the basic i/s/o relation (1.1) can be rewritten in the form

$$\begin{bmatrix} x(n+1) \\ x(n) \\ w(n) \end{bmatrix} \in V, \quad n \in \mathbb{Z}^+ = \{0, 1, 2, \ldots\}, \ x(0) = x_0, \tag{3.1}$$

where V is the subspace of $\mathfrak{K} := \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$ given by

$$V = \left\{ \left[\begin{smallmatrix} z \\ x \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right] \;\middle|\; \begin{matrix} z = Ax + Bu, \\ y = Cx + Du, \end{matrix} \quad w = \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right], \ x \in \mathcal{X}, \ u \in \mathcal{U} \right\}. \tag{3.2}$$

It is not difficult to show that the subspace V obtained in this way has the following four properties:

(i) V is closed in $\mathfrak{K}$.

(ii) For every $x \in \mathcal{X}$ there is some $\left[\begin{smallmatrix} z \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$ such that $\left[\begin{smallmatrix} z \\ x \\ w \end{smallmatrix}\right] \in V$.

(iii) If $\left[\begin{smallmatrix} z \\ 0 \\ 0 \end{smallmatrix}\right] \in V$, then $z = 0$.

(iv) The set $\left\{ \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right] \;\middle|\; \left[\begin{smallmatrix} z \\ x \\ w \end{smallmatrix}\right] \in V \text{ for some } z \in \mathcal{X} \right\}$ is closed in $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$.

Definition 3.1. A triple $\Sigma = (V; \mathcal{X}, \mathcal{W})$, where the (*internal*) *state space* $\mathcal{X}$ is a Hilbert space and the (*external*) *signal space* $\mathcal{W}$ is a Kreĭn space and V is a subspace

[8] Both [BS05] and [AS06a] contain short sections on the geometry of a Kreĭn space. For more detailed treatments we refer the reader to one of the books [ADRdS97], [AI89] and [Bog74].

of the product space $\mathfrak{K} := \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix}$ is called a *s/s* (*state/signal*) *node* if it has properties (i)–(iv) listed above. We interpret $\mathfrak{K}$ as a Kreĭn space with the inner product

$$\left[\!\!\left[\begin{bmatrix} z \\ x \\ w \end{bmatrix}, \begin{bmatrix} z' \\ x' \\ w' \end{bmatrix} \right]\!\!\right]_{\mathfrak{K}} = -\langle z, z' \rangle_{\mathcal{X}} + \langle x, x' \rangle_{\mathcal{X}} + [w, w']_{\mathcal{W}}, \quad \begin{bmatrix} z \\ x \\ w \end{bmatrix}, \begin{bmatrix} z' \\ x' \\ w' \end{bmatrix} \in \mathfrak{K}, \tag{3.3}$$

and we call $\mathfrak{K}$ the *node space* and V the *generating subspace*.

By a *trajectory* of Σ we mean a pair of sequences (x, w) satisfying (3.1). We call x the *state component* and w the *signal component* of this trajectory. By the *s/s system* Σ we mean the s/s node Σ together with all its trajectories.

The conditions (i)–(iv) above have natural interpretations in terms of the trajectories of Σ: for each $x_0 \in \mathcal{X}$ condition (ii) gives forward existence of at least one trajectory (x, w) of Σ with $x(0) = x_0$. Condition (iii) implies that a trajectory (x, w) is determined uniquely by x_0 and w, and conditions (i) and (iv) imply that the signal part x depends continuously in $\mathcal{X}^{\mathbb{Z}^+}$ on $x_0 \in \mathcal{X}$ and $w \in \mathcal{W}^{\mathbb{Z}^+}$.

A s/s system Σ is *controllable* if the set of all states $x(n)$, $n \geq 1$, which appear in some trajectory (x, w) of Σ with $x(0) = 0$ (i.e., an *externally generated trajectory*) is dense in $\mathcal{X}$. The system Σ is *observable* if there do not exist any nontrivial trajectories (x, w) where the signal component w is identically zero. Finally, Σ is *minimal* if Σ is both controllable and observable.

Above we explained how to interpret an i/s/o system $\Sigma_{\mathrm{i/s/o}}$ as a s/s system. Conversely, *from every s/s system* Σ *it is possible to create not only one, but infinitely many i/s/o systems*. The representation (3.2) is characterized by the fact that it is a *graph representation of* V *over* $\begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}$ where $\mathcal{U}$ is one of the two components in a direct sum decomposition of $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ (not necessarily orthogonal) of $\mathcal{W}$. Indeed, splitting w into $w = \begin{bmatrix} y \\ u \end{bmatrix}$ and reordering the components we find that (3.2) is equivalent to

$$V = \left\{ \begin{bmatrix} z \\ y \\ x \\ u \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \\ \mathcal{X} \\ \mathcal{U} \end{bmatrix} \,\middle|\, \begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}, \ \begin{bmatrix} x \\ u \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \right\}. \tag{3.4}$$

As shown in [AS05], *the generating subspace of every s/s system* Σ *has at least one* (*hence infinitely many*) *graph representation of this type*. A direct sum decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ of $\mathcal{W}$ is called an *admissible i/o* (*input/output*) *decomposition of* $\mathcal{W}$ *for* Σ, or simply an admissible decomposition, if it leads to a graph representation of the generating subspace of Σ described above. From each such graph representation of V we get an i/s/o system $\Sigma_{\mathrm{i/s/o}} = \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}; \mathcal{U}, \mathcal{X}, \mathcal{Y} \right)$ of Σ, which we call an *i/s/o representation* of Σ.

The above definitions are taken from [AS05], [AS06a], and [AS06b]. It turns out that a very large part of the proof of the H-passivity theory covered in Section 2 can be carried out directly in the s/s setting, rather than applying the same arguments separately with the scattering, impedance, and transmission supply rates. This leads to both a simplification and to a unification of the whole theory. Below we present the most basic parts of the H-passive s/s theory, and refer the reader to [AS05]–[AS06c] for details.

Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a s/s node. The *adjoint* $\Sigma_* = (V_*; \mathcal{X}, \mathcal{W}_*)$ of Σ (introduced in [AS06a, Section 4]) is another s/s node, with the same state space $\mathcal{X}$ as Σ, and with the signal space $\mathcal{W}_* = -\mathcal{W}$.[9] The generating subspace V_* of Σ_* is given by

$$V_* = \left\{ \begin{bmatrix} x_* \\ z_* \\ w_* \end{bmatrix} \;\middle|\; \begin{bmatrix} z_* \\ x_* \\ w_* \end{bmatrix} \in V^{[\perp]} \right\},$$

where $V^{[\perp]}$ is the orthogonal companion to V with respect to the Kreĭn space inner product of $\mathfrak{K}$.[10] The adjoint system Σ_* is determined by the property that

$$-\langle x(n+1), x_*(0)\rangle_{\mathcal{X}} + \langle x(0), x_*(n+1)\rangle_{\mathcal{X}} + \sum_{k=0}^{n} [w(k), w_*(n-k)]_{\mathcal{W}} = 0, \quad n \in \mathbb{Z}^+,$$

for all trajectories (x, w) of Σ.

The following definition is the s/s version of Definitions 1.1 and 1.2.

Definition 3.2. Let H be a positive self-adjoint operator in the Hilbert space $\mathcal{X}$. A s/s system Σ is

(i) *forward H-passive* if $x(n) \in \mathcal{D}(\sqrt{H})$ and

$$\|\sqrt{H}x(n+1)\|^2_{\mathcal{X}} - \|\sqrt{H}x(n)\|^2_{\mathcal{X}} \le [w(n), w(n)]_{\mathcal{W}}, \quad n \in \mathbb{Z}^+,$$

for every trajectory (x, w) of Σ with $x(0) \in \mathcal{D}(\sqrt{H})$,

(ii) *forward H-conservative* if the above inequality holds as an equality,

(iii) *backward H-passive* or *H-conservative* if Σ_* is forward H^{-1}-passive or H^{-1}-conservative, respectively,

(iv) *H-passive* or *H-conservative* if it is both forward and backward H-passive or H-conservative, respectively,

(v) *passive* or *conservative* if it is $1_{\mathcal{X}}$-passive or $1_{\mathcal{X}}$-conservative.

To formulate a s/s version of Theorems 2.1, 2.2 and 2.3 we need a s/s analogue of the transfer function of an i/s/o system. Such an analogue is most easily obtained in the time domain (as opposed to the frequency domain), and it amounts to the introduction of a *behavior*[11] *on the signal space* $\mathcal{W}$. By this we mean a closed right-shift invariant subspace of the Fréchet space $\mathcal{W}^{\mathbb{Z}^+}$. Thus, in particular, the set $\mathfrak{W}$ of all sequences w

[9] Algebraically $-\mathcal{W}$ is the same space as $\mathcal{W}$, but the inner product in $-\mathcal{W}$ is obtained from the one in $\mathcal{W}$ by multiplication by the constant factor -1.

[10] Thus, $V^{[\perp]} = \{k_* \in \mathfrak{K} \mid [k, k_*]_{\mathfrak{K}} = 0 \text{ for all } k \in V\}$. Note that V_* differs from $V^{[\perp]}$ only by the order of the first two components.

[11] Our behaviors are what Polderman and Willems call *linear time-invariant mainfest behaviors* in [PW98, Definitions 1.3.4, 1.4.1, and 1.4.2]. We refer the reader to this book for further details on behaviors induced by systems with a finite-dimensional state space and for an account of the extensive literature on this subject.

that are the signal parts of externally generated trajectories of a given s/s system Σ is a behavior. We call this the *behavior induced by* Σ, and refer to Σ as a *s/s realization of* $\mathfrak{W}$, or, in the case where Σ is minimal, as a *minimal s/s realization of* $\mathfrak{W}$. A behavior is *realizable* if it has a s/s realization.

Two s/s systems Σ_1 and Σ_2 with the same signal space are *externally equivalent* if they induce the same behavior. This property is related to the notion of *pseudo-similarity*. Two s/s systems $\Sigma = (V; \mathcal{X}, \mathcal{W})$ and $\Sigma_1 = (V_1; \mathcal{X}_1, \mathcal{W})$ are called *pseudo-similar* if there exists an injective densely defined closed linear operator $R\colon \mathcal{X} \to \mathcal{X}_1$ with dense range such that the following conditions hold:

> If $(x(\cdot), w(\cdot))$ is a trajectory of Σ on $\mathbb{Z}^+$ with $x(0) \in \mathcal{D}(R)$, then $x(n) \in \mathcal{D}(R)$ for all $n \in \mathbb{Z}^+$ and $(Rx(\cdot), w(\cdot))$ is a trajectory of Σ_1 on $\mathbb{Z}^+$, and conversely, if $(x_1(\cdot), w(\cdot))$ is a trajectory of Σ_1 on $\mathbb{Z}^+$ with $x_1(0) \in \mathcal{R}(R)$, then $x_1(n) \in \mathcal{R}(R)$ for all $n \in \mathbb{Z}^+$ and $(R^{-1}x_1(\cdot), w(\cdot))$ is a trajectory of Σ on $\mathbb{Z}^+$.

In particular, if Σ_1 and Σ_2 are pseudo-similar, then they are externally equivalent. Conversely, if Σ_1 and Σ_2 are minimal and externally equivalent, then they are necessarily pseudo-similar. Moreover, a realizable behavior $\mathfrak{W}$ on the signal space $\mathcal{W}$ has a minimal s/s realization, which is determined uniquely by $\mathfrak{W}$ up to pseudo-similarity. (See [AS05, Section 7] for details.)

The *adjoint* of the behavior $\mathfrak{W}$ on $\mathcal{W}$ is a behavior $\mathfrak{W}_*$ on $\mathcal{W}_*$ defined as the set of sequences w_* satisfying

$$\sum_{k=0}^{n}[w(k), w_*(n-k)]_{\mathcal{W}} = 0, \quad n \in \mathbb{Z}^+,$$

for all $w \in \mathfrak{W}$. If $\mathfrak{W}$ is induced by Σ, then $\mathfrak{W}_*$ is (realizable and) induced by Σ_*, and the adjoint of $\mathfrak{W}_*$ is the original behavior $\mathfrak{W}$.

The following definition is a s/s analogue of our earlier definitions of the Schur, Carathéodory, and Potapov classes of transfer functions.

Definition 3.3. A behavior $\mathfrak{W}$ on $\mathcal{W}$ is

(i) *forward passive* if

$$\sum_{k=0}^{n}[w(k), w(k)]_{\mathcal{W}} \geq 0, \quad w \in \mathfrak{W},\ n \in \mathbb{Z}^+,$$

(ii) *backward passive* if $\mathfrak{W}_*$ is forward passive,

(iii) *passive* if it is realizable[12] and both forward and backward passive.

It is not difficult to see that a s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is forward H-passive if and only if $H > 0$ is a solution of the generalized s/s KYP (Kalman–Yakubovich–

[12] We do not know if the realizability assumption is redundant or not.

Popov) inequality[13]

$$\|\sqrt{H}z\|^2_{\mathcal{X}} - \|\sqrt{H}x\|^2_{\mathcal{X}} \le [w, w]_{\mathcal{W}}, \quad \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V,\ x \in \mathcal{D}(\sqrt{H}), \tag{3.5}$$

and that it is forward H-conservative if and only if the above inequality holds as an equality.

The following proposition is a s/s version of parts (i) of Theorems 2.1, 2.2, and 2.3.

Proposition 3.4. *Let $\mathfrak{W}$ be the behavior induced by a s/s system Σ.*

(i) *If Σ is forward H-passive for some $H > 0$, then $\mathfrak{W}$ is forward passive.*

(ii) *If Σ is backward H-passive for some $H > 0$, then $\mathfrak{W}$ is backward passive.*

(iii) *If Σ is forward H_1-passive for some $H_1 > 0$ and backward H_2-passive for some $H_2 > 0$, then Σ is both H_1-passive and H_2-passive, and $\mathfrak{W}$ is passive.*

The following theorem generalizes parts (ii) of Theorems 2.1, 2.2, and 2.3.

Theorem 3.5. *Let $\mathfrak{W}$ be a passive behavior on $\mathcal{W}$. Then*

(i) *$\mathfrak{W}$ has a minimal passive s/s realization.*

(ii) *Every H-passive realization Σ of $\mathfrak{W}$ is pseudo-similar to a passive realization Σ_H with pseudo-similarity operator $\sqrt{H}$. The system Σ_H is determined uniquely by Σ and H.*

(iii) *Every minimal realization of $\mathfrak{W}$ is H-passive for some $H > 0$, and it is possible to choose H in such a way that the system Σ_H in* (ii) *is minimal.*

Assertion (ii) can be interpreted in the following way: we can always convert an H-passive s/s system into a passive one by simply replacing the original norm $\|\cdot\|_{\mathcal{X}}$ in the state space by the new norm $\|x\|_H = \|\sqrt{H}x\|_{\mathcal{X}}$, which is finite for all $x \in \mathcal{D}(\sqrt{H})$, and then completing $\mathcal{D}(\sqrt{H})$ with respect to this new norm.

We shall end this section with a result that says that a suitable subclass of all operators $H > 0$ for which a s/s system Σ is H-passive can be partially ordered. Here we use the following partial ordering of nonnegative self-adjoint operators on $\mathcal{X}$: if H_1 and H_2 are two nonnegative self-adjoint operators on the Hilbert space $\mathcal{X}$, then we write $H_1 \preceq H_2$ whenever $\mathcal{D}(H_2^{1/2}) \subset \mathcal{D}(H_1^{1/2})$ and $\|H_1^{1/2}x\| \le \|H_2^{1/2}x\|$ for all $x \in \mathcal{D}(H_2^{1/2})$. For *bounded* nonnegative operators H_1 and H_2 with $\mathcal{D}(H_2) = \mathcal{D}(H_1) = \mathcal{X}$ this ordering coincides with the standard ordering of bounded self-adjoint operators.

For each s/s system Σ we denote the set of operators $H > 0$ for which Σ is H-passive by M_Σ, and we let $M_\Sigma^{\min}$ be the set of $H \in M_\Sigma$ for which the system Σ_H in assertion (ii) of Theorem 3.5 is minimal.

[13] In particular, in order for the first term in this inequality to be well-defined we require $z \in \mathcal{D}(\sqrt{H})$ whenever $\begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V$ and $x \in \mathcal{D}(\sqrt{H})$.

Theorem 3.6. *Let Σ be a minimal s/s system with a passive behavior. Then $M_\Sigma^{\min}$ contains a minimal element $H_\circ$ and a maximal element $H_\bullet$, i.e., $H_\circ \preceq H \preceq H_\bullet$ for every $H \in M_\Sigma^{\min}$.*

The two extremal storage functions $E_{H_\circ}$ and $E_{H_\bullet}$ correspond to Willems' [Wil72a], [Wil72b] *available storage* and *required supply*, respectively (there presented in an i/s/o setting). In the terminology of Arov [Aro79b], [Aro95], [Aro99] (likewise in an i/s/o setting), $\Sigma_{H_\circ}$ is the *optimal* and $\Sigma_{H_\bullet}$ is the $*$-*optimal* realization of $\mathfrak{W}$.

4. Scattering, impedance and transmission representations of s/s systems

The results presented in Section 2 can be recovered from those in Section 3, together with a number of additional results. This is done by studying different i/s/o representations of a s/s system. Depending on the admissible i/o decomposition of the signal space $\mathcal{W}$ into an input space $\mathcal{U}$ and an output space $\mathcal{Y}$ we get different supply rates (inherited from the Kreĭn space inner product in $\mathcal{W}$).

Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a s/s system, and decompose $\mathcal{W}$ into the direct sum of an input space $\mathcal{U}$ and an output space $\mathcal{Y}$. Furthermore, suppose that this decomposition is admissible, so that it gives rise to an i/s/o representation $\Sigma_{\mathrm{i/s/o}}$ of Σ. In the case of a *fundamental decomposition* $\mathcal{W} = -\mathcal{Y} \,[+]\, \mathcal{U}$, where $\mathcal{Y}$ and $\mathcal{U}$ are Hilbert spaces (i.e., $-\mathcal{Y}$ is an anti-Hilbert space) and $-\mathcal{Y}$ and $\mathcal{U}$ are orthogonal in $\mathcal{W}$, the inner product in $\mathcal{W}$ is given by

$$\left[\begin{bmatrix} y \\ u \end{bmatrix}, \begin{bmatrix} y' \\ u' \end{bmatrix}\right]_{\mathcal{W}} = -\langle y, y'\rangle_{\mathcal{Y}} + \langle u, u'\rangle_{\mathcal{U}},$$

which leads to a *scattering supply rate* for the i/s/o representation $\Sigma_{\mathrm{i/s/o}}$. In this case we call $\Sigma_{\mathrm{i/s/o}}$ an *admissible scattering representation* of Σ. In the case of a (nonorthogonal) *Lagrangian decomposition*, where both $\mathcal{Y}$ and $\mathcal{U}$ are Lagrangian[14] subspaces of $\mathcal{W}$ we get an *impedance supply rate* and an *admissible impedance representation* of Σ. Finally, if $\mathcal{W} = -\mathcal{Y} \,[\dot{+}]\, \mathcal{U}$ is an arbitrary *orthogonal decomposition* of $\mathcal{W}$ (not necessarily fundamental), then we get a *transmission supply rate* and an *admissible transmission representation* of Σ. Thus, in the s/s setting the external Cayley transform and the Potapov–Ginzburg transform that we presented in Section 2 are simply two different ways at looking at the same s/s system, via different i/o decompositions of the signal space $\mathcal{W}$ into an input space $\mathcal{U}$ and an output space $\mathcal{Y}$.

The following proposition is related to the discussions at the beginning of Sections 2.1 and 2.2.

Proposition 4.1. *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a forward H-passive s/s system for some $H > 0$. Then the following claims hold.*

[14] A subspace of a Kreĭn space is Lagrangian if it coincides with its own orthogonal companion.

(i) Σ *is* H*-passive if and only if* Σ *has an admissible scattering representation, in which case* every *fundamental decomposition of* $\mathcal{W}$ *is admissible.*

(ii) *If* Σ *has an admissible impedance representation, then* Σ *is* H*-passive.*

The converse of (ii) is not true: there do exist passive s/s systems which do not have any admissible impedance representation, even if we require the positive and negative dimensions of $\mathcal{W}$ to be the same. Every H-passive s/s system does have some admissible transmission representations (for example, every scattering representation can be interpreted as a transmission representation), but in general there also exist orthogonal decompositions of the signal space that are not admissible.

One way to prove many of the results listed above is to pass to some particular i/s/o representation $\Sigma_{\mathrm{i/s/o}}$ of the s/s system Σ, to prove the corresponding result for $\Sigma_{\mathrm{i/s/o}}$, and to reinterpret the result for the s/s system Σ. In many cases the most convenient choice is to use a scattering representation, corresponding to some admissible fundamental decomposition of the signal space. We recall from Proposition 4.1 that if Σ is H-passive for some $H > 0$, then every fundamental decomposition is admissible. However, this is not the only possible choice. If $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ is an arbitrary admissible i/o decomposition for Σ, then Σ is forward or backward H-passive if and only if the corresponding i/s/o system $\Sigma_{\mathrm{i/s/o}}$ is forward or backward H-passive with respect to the supply rate on $\mathcal{Y} \dotplus \mathcal{U}$ inherited from the inner product $[\cdot,\cdot]_{\mathcal{W}}$. Thus, in the family of i/s/o systems $\Sigma_{\mathrm{i/s/o}} = \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{U}, \mathcal{X}, \mathcal{Y}\right)$ that we get from Σ by varying the i/o decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ the coefficients $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ vary, and so do the supply rates $j(u, y)$, but *the set of solutions of the generalized KYP inequalities* (1.6) *and* (1.10) *stay the same*.

Up to now we have only considered *admissible* i/o decompositions of the signal space $\mathcal{W}$ of a s/s system Σ. As we commented earlier, not every Lagrangian or orthogonal decomposition need be admissible for Σ, even if Σ is H-passive for some $H > 0$. However, it is still possible to study also these non-admissible decompositions by replacing the i/s/o representations by *left or right affine representations* of Σ. These are defined for arbitrary decompositions $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ (not only for the admissible ones). By a *right affine i/s/o representation of* Σ we mean an i/s/o system[15]

$$\Sigma^r_{\mathrm{i/s/o}} = \left(\left[\begin{array}{c|c} A' & B' \\ \hline C'_{\mathcal{Y}} & D'_{\mathcal{Y}} \\ C'_{\mathcal{U}} & D'_{\mathcal{U}} \end{array}\right]; \mathcal{L}, \mathcal{X}, \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix}\right)$$

with the following two properties: 1) $D' = \begin{bmatrix} D'_{\mathcal{Y}} \\ D'_{\mathcal{U}} \end{bmatrix}$ has a bounded left-inverse, and 2) $\left(x, \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]\right)$ is a trajectory of Σ if and only if $\left(\ell, x, \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]\right)$ is a trajectory of $\Sigma^r_{\mathrm{i/s/o}}$ for some sequence ℓ with values in $\mathcal{L}$. By a *left affine i/s/o representation of* Σ we mean

[15] Here the new input space $\mathcal{L}$ is an auxiliary Hilbert space called the *driving variable* space.

an i/s/o system[16]

$$\Sigma^l_{\mathrm{i/s/o}} = \left(\left[\begin{array}{c|cc} A'' & B''_{\mathcal{Y}} & B''_{\mathcal{U}} \\ \hline C'' & D''_{\mathcal{Y}} & D''_{\mathcal{U}} \end{array} \right]; \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix}, \mathcal{X}, \mathcal{K} \right)$$

with the following two properties: 1) $D'' = \begin{bmatrix} \mathfrak{D}''_{\mathcal{Y}} & \mathfrak{D}''_{\mathcal{U}} \end{bmatrix}$ has a bounded right-inverse, and 2) $\left(x, \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]\right)$ is a trajectory of Σ if and only if $\left(\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right], x, 0\right)$ is a trajectory of $\Sigma^l_{\mathrm{i/s/o}}$ (i.e., the output is identically zero in $\mathcal{K}$). The transfer functions of these systems are called the right, respectively left, affine transfer functions of Σ corresponding to the i/o decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$. Note, in particular, that the right and left affine transfer functions are now decomposed into $\mathfrak{D}' = \begin{bmatrix} \mathfrak{D}'_{\mathcal{Y}} \\ \mathfrak{D}'_{\mathcal{U}} \end{bmatrix}$ and $\mathfrak{D}'' = \begin{bmatrix} \mathfrak{D}''_{\mathcal{Y}} & \mathfrak{D}''_{\mathcal{U}} \end{bmatrix}$, respectively.

Let

$$\Omega(\Sigma^r_{\mathrm{i/s/o}}) = \{z \in \Lambda_{A'} \mid \mathfrak{D}'_{\mathcal{U}}(z) \text{ has a bounded inverse}\},$$
$$\Omega(\Sigma^l_{\mathrm{i/s/o}}) = \{z \in \Lambda_{A''} \mid \mathfrak{D}''_{\mathcal{Y}}(z) \text{ has a bounded inverse}\},$$

and let

$\Omega^r(\Sigma; \mathcal{U}, \mathcal{Y})$ be the union of the above sets $\Omega(\Sigma^r_{\mathrm{i/s/o}})$,

$\Omega^l(\Sigma; \mathcal{U}, \mathcal{Y})$ be the union of the above sets $\Omega(\Sigma^l_{\mathrm{i/s/o}})$.

We can now define the notions of right and left generalized transfer functions of Σ with input space $\mathcal{U}$ and output space $\mathcal{Y}$ on the sets $\Omega^r(\Sigma; \mathcal{U}, \mathcal{Y})$ and $\Omega^l(\Sigma; \mathcal{U}, \mathcal{Y})$, respectively, by the formulas

$$\mathfrak{D}_r(z) = \mathfrak{D}'_{\mathcal{Y}}(z)\mathfrak{D}'_{\mathcal{U}}(z)^{-1}, \tag{4.1}$$

$$\mathfrak{D}_l(z) = -\mathfrak{D}''_{\mathcal{Y}}(z)^{-1}\mathfrak{D}''_{\mathcal{U}}(z), \tag{4.2}$$

respectively.

Theorem 4.2. *The right-hand side of* (4.1) *does not depend on the choice of* $\Sigma^r_{\mathrm{i/s/o}}$ *as long as* $\Omega(\Sigma^r_{\mathrm{i/s/o}}) \ni z$, *and the right-hand side of* (4.2) *does not depend on the choice of* $\Sigma^l_{\mathrm{i/s/o}}$ *as long as* $\Omega(\Sigma^l_{\mathrm{i/s/o}}) \ni z$.

Theorem 4.3. *The right and left generalized transfer functions defined by* (4.1) *and* (4.2), *respectively, coincide on*

$$\Omega(\Sigma; \mathcal{U}, \mathcal{Y}) = \Omega^r(\Sigma; \mathcal{U}, \mathcal{Y}) \cap \Omega^l(\Sigma; \mathcal{U}, \mathcal{Y})$$

(*whenever this set is nonempty*). *If the i/o decomposition* $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ *is admissible, and if* A *is the main operator of the corresponding i/s/o representation of* Σ, *then*

$$\Omega^r(\Sigma; \mathcal{U}, \mathcal{Y}) = \Omega^l(\Sigma; \mathcal{U}, \mathcal{Y}) = \Lambda_A,$$

and the left and right generalized transfer functions coincide with the ordinary transfer function corresponding to the decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$

[16] Here the new output space $\mathcal{K}$ is an auxiliary Hilbert space called the *error variable* space.

In the case where the s/s system Σ is H-passive for some $H > 0$ we can say more. In this case it is possible to choose the different affine representations of Σ in such a way that the right and left transfer functions are defined in the whole unit disk $\mathbb{D}$ and belong to H^∞, and they will even be *right and left coprime in* H^∞, respectively. In this way we obtain right and left coprime transmission representations of Σ, and in the case that the positive and negative dimensions of the signal space $\mathcal{W}$ are the same we also obtain right and left coprime impedance representations. The corresponding right and left coprime affine transfer functions will be generalized Potapov and Carathéodory class functions, respectively.

5. Further extensions

The results of Sections 3 and 4 are taken primarily from [AS05], [AS06a]–[AS06c]. At present they do not yet make up a complete theory that would be ready to replace the classical i/s/o theory. However, the following additional *discrete part* ingredients of the s/s theory are presently under active development:

- The study of the *interconnection* of two s/s systems (this is the s/s analogue of feedback).
- *Lossless behaviors* and *bi-lossless extensions* of passive behaviors (including the s/s analogue of Darlington synthesis).
- Additional *representations* of generalized Carathéodory and Potapov class functions.
- External and internal *symmetry* of s/s systems (including reciprocal systems).
- Further studies of the *stability properties* of passive s/s systems.
- Conditions for *ordinary similarity* (as opposed to pseudo-similarity) of minimal passive realizations.

An even larger project is still in its infancy, namely the *extension of the s/s theory to continuous time systems*. Some preliminary results in this direction have been obtained in [BS05] and [MS06a], [MS06b].

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