Control and numerical approximation of the wave and heat equations

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Abstract. In recent years important progress have been done in the context of numerical approximation of controllability problems for PDEs. It is by now well known that, often, numerical approximation schemes that are stable for solving initial-boundary value problems, develop instabilities when applied to controllability problems. This is due to the presence of spurious high frequency numerical solutions that the control mechanisms are not able to control uniformly as the mesh-size tends to zero. However, the theory is far from being complete. In this article we present some new results in this framework for the wave and the heat equations, which also raise a number of open questions and future directions of research. We first prove that a two-grid method, introduced by R. Glowinski, that is by now well-known to guarantee convergence for the $1-d$ wave equation, also converges in the semilinear setting for globally Lipschitz nonlinearities. This result provides a further evidence of the robustness of the two-grid method. We then show that boundary controls for finite-difference space semi-discretizations of the heat equation converge when applied all along the boundary of the domain, a fact that does not hold for wave-like equations. This confirms that the strong irreversibility of the heat equation enhances the control properties of its numerical approximation schemes. This result fails when the control is restricted to some subsets of the boundary because of the lack of unique continuation of some high frequency eigenvectors of the underlying discrete eigenvalue problem.

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1. Introduction

In recent years important progresses have been done in the context of numerical approximation of controllability problems for PDEs. It is by now well known that, often, numerical approximation schemes that are stable for solving an initial-boundary value problem, develop instabilities when applied to controllability problems. This is due to the presence of spurious high frequency numerical solutions that the control mechanisms are not able to control uniformly as the mesh-size tends to zero.

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To cure these instabilities a number of methods have been introduced in the literature. We refer to [30] for a recent survey article on the topic.

In this context and in an effort to build a general theory, there are two prototypical equations that need to be understood first of all: the *wave equation* and the *heat equation*.

In the framework of the linear wave equation, R. Glowinski [6] introduced a two-grid control mechanism that allows filtering the high frequency numerical spurious solutions and guarantee the convergence of controls. There are clear numerical evidences of the convergence of the method whose proof has been successfully carried out in [17] in the $1-d$ case by using discrete multipliers. More recently the same result has been proved, with a better estimate on the minimal control time, in [16] by using Ingham type inequalities. Other methods have also been developed for avoiding these instabilities to occur: Tychonoff regularization, Fourier filtering, mixed finite elements,...(see [30]). But most of the existing theory is devoted to linear problems. The first part of this article is devoted to show how the convergence result of the two-grid algorithm can be extended to semilinear systems too, with globally Lipschitz nonlinearities. This result adds one more evidence of the robustness and efficiency of the two-grid algorithm for the control of wave problems.

The high frequency spurious numerical solutions for the wave equation are due to the existence of wave-packets that travel with a vanishing group velocity (see [21], [30]). This can be understood by analyzing the symbol of the operator and the dynamics of the Hamiltonian system generating the bicharacteristic rays. However, one expects that the heat equation, because of its intrinsic time-irreversibility and strong damping should escape to those pathologies and that most common numerical approximation schemes should be controllable, uniformly with respect to the mesh-size. This holds indeed in the $1-d$ setting (see [14]). But, surprisingly enough, this property may fail to hold in $2-d$ even for the simplest finite-difference semi-discretization scheme for the heat equation in the square. This is due to the fact that there are some high-frequency numerical solutions that do not fulfill the classical property of unique continuation of the continuous heat equation. Thus, at the control level, the numerical approximation schemes may generate some solutions which are insensitive to the action of controls. Strictly speaking this happens when the control acts on some (small enough) subsets of the boundary where the equation holds. However, this fact clearly indicates a major difference in the control theoretical behavior of the continuous and the semi-discrete heat equation since the first one is controllable from any open and non-empty subset of the boundary while the second one is not. Characterizing completely the subsets of the boundary for which these pathologies arise is probably a difficult problem. In this article we prove that convergence occurs when the controls act everywhere on the boundary of the domain. This confirms that heat equations are better behaved than wave ones. Indeed, for the wave equation, even if controls act everywhere on the boundary of the domain, the uniform controllability property for numerical approximation schemes may fail because of the existence of spurious numerical solutions that are trapped in the interior of the mesh without reaching the boundary in an uniform time. Our positive result for the heat equation shows that this kind of spurious solutions are ruled out due to the strong dissipativity of the heat equation and its numerical approximation schemes but, so far, only when the control is distributed everywhere on the boundary.

The lack of unique continuation for semi-discrete heat equations is due to the fact that the property fails to hold for the spectrum of the discrete Laplacian. Indeed, for the Dirichlet spectrum of the continuous Laplacian, unique continuation holds in the sense that, when the normal derivative of an eigenfunction vanishes in a subset of the boundary, the eigenfunction vanishes everywhere. This property fails to hold for the eigenvectors of the discrete Laplacian. The main method to prove unique continuation properties in the continuous framework are the so-called Carleman inequalities. But the discrete analogue is still to be developed. An alternate natural way of addressing this issue, in the spirit of the classical theory of numerical analysis, would consist in viewing the solutions of the discrete problem as a perturbation of those of the continuous one and applying the continuous Carleman inequalities. This approach has been successfully applied in [31] to elliptic equations with irregular coefficients in the principal part. Developing this program in the context of discrete elliptic equations is an interesting open problem.

The two topics we address in this article also raise a number of interesting open problems and future directions of research that we mention briefly. Some of them, in our opinion, are deep and will require important research efforts. The interest of these problems goes much beyond Control Theory since they mainly concern the way classical numerical analysis and the existing theory of partial differential equations have to be melt to address subtle qualitative aspects of numerical solutions. We hope that this article will serve to stimulate research in this area.

2. Controllability of the two-grid approximation scheme for the $1-d$ semilinear wave equation

One of the main drawbacks of the existing theory to analyze the controllability of numerical approximation schemes for PDE is that it often relies on Fourier analysis. This makes it of little use for nonlinear problems. However there is by now an extensive literature on the controllability of semilinear PDE and, in particular, of wave and heat equations. Therefore, it is natural to develop numerical methods allowing to address these nonlinear models and to build convergent numerical schemes for their control.

In this section we consider the $1-d$ semilinear wave equation with boundary control:

$$\begin{cases} y_{tt} - y_{xx} + f(y) = 0, & x \in (0,\,1),\ 0 < t < T, \\ y(0,\,t) = 0, \quad y(1,\,t) = v(t), & 0 < t < T, \\ y(x,\,0) = y^0(x),\ y_t(x,\,0) = y^1(x), & x \in (0,\,1). \end{cases} \tag{2.1}$$

Here the control $v = v(t)$ enters into the system through the extreme $x = 1$ of the boundary.

This semilinear wave equation is known to be controllable under sharp growth conditions on the nonlinearity. Namely, if

$$|f'(s)| \leq C \log^2(1 + |s|) \quad \text{for all } s \in \mathbb{R} \tag{2.2}$$

for some $C > 0$, system (2.1) is exactly controllable in any time $T > 2$. This means that for all $(y^0, y^1) \in L^2(0, 1) \times H^{-1}(0, 1)$ and $(z^0, z^1) \in L^2(0, 1) \times H^{-1}(0, 1)$ there exists a control $v \in L^2(0, T)$ such that the solution y of (2.1) satisfies

$$y(x, T) = z^0(x), \quad y_t(x, T) = z^1(x) \quad \text{in } (0, 1). \tag{2.3}$$

This result was proved in [26] for $C > 0$ sufficiently small in (2.2) and, without restrictions on the size of the constant C, in [1].

This growth condition is sharp since blow-up phenomena may occur for nonlinearities growing faster at infinity and, due to the finite speed of propagation, boundary controls are unable to avoid blow-up to occur. In that case controllability fails.

The most common method to derive the exact controllability property of semilinear equations is based on the following ingredients:

- a fixed point argument;
- sharp estimates on the dependence of controls for the underlying linear equation perturbed by a potential.

We refer to [25] where this method was introduced in the context of the wave equation (see also [22] for further developments, and [23] for an updated survey on this problem) and to [4] where the same technique was applied to semilinear heat equations.

Knowing that the semilinear wave equation (2.1) is controllable under the growth condition (2.2) it is natural to analyze whether the controls can be obtained as limits of controls of numerical approximation schemes. As we have explained in the introduction this issue is delicate even for linear problems, and it is necessarily more complex for nonlinear ones.

Among the possible remedies to the lack of convergence of the standard conservative schemes the two-grid method introduced in [6] seems to be the one that is better adapted to semilinear problems. In this section we confirm this assertion by proving its convergence in this nonlinear setting for globally Lipschitz nonlinearities.

The two-grid scheme is, roughly, as follows.

Given an integer $N \in \mathbb{N}$ we introduce the partition $\{x_j = jh\}_{j=0,\dots,N+1}$ of the interval $(0, 1)$ with $h = 1/(N + 1)$ so that $x_0 = 0$ and $x_{N+1} = 1$.

We then consider the conservative finite-difference semi-discretization of the semilinear wave equation (2.1) as follows:

$$\begin{cases} y_j'' + \frac{2y_j - y_{j+1} - y_{j-1}}{h^2} + f(y_j) = 0, & j = 1, \dots, N,\ 0 < t < T, \\ y_0(t) = 0,\ y_{N+1}(t) = v(t), & 0 < t < T, \\ y_j(0) = y_j^0,\ y_j'(0) = y_j^1, & j = 0, \dots, N + 1. \end{cases} \tag{2.4}$$

The scheme is conservative in the sense that, in the absence of control (i.e. for $v \equiv 0$) the energy of solutions is conserved. The same property holds for the continuous version (2.1). In that case the energy is given by

$$E(t) = \frac{1}{2}\int_0^1 \left[y_t^2(x,t) + y_x^2(x,t)\right] dx + \int_0^1 F(y(x,t))\, dx,$$

where F is a primitive of f, i.e. $F(z) = \int_0^z f(s)ds$. In the semi-discrete case the corresponding energy is

$$E_h(t) = \frac{h}{2}\sum_{j=0}^{N}\left[|y_j'|^2 + \left|\frac{y_{j+1}-y_j}{h}\right|^2\right] + h\sum_{j=0}^{N} F(y_j).$$

The goal of this section is to analyze the controllability of (2.4) and whether, as $h \to 0$, the controls of (2.4) converge to those of (2.1). The controls being, in general, non unique, one has to be precise when discussing their convergence. Here, in the linear context, we shall always refer to the controls of minimal $L^2(0,T)$-norm which are given by the so called Hilbert Uniqueness Method (HUM) ([13]). As we mentioned above, in the nonlinear case, the controls we shall deal with are obtained by fixed point methods on the basis of the HUM controls for the linearized problems.

But, even in the linear case, to guarantee convergence as $h \to 0$, the final control requirement has to be relaxed, or the numerical scheme modified.

In [29] it was proved that, if the exact controllability condition is relaxed to the approximate controllability one (in which the state is required to reach an ε-neighborhood of the target), then convergence occurs in the linear framework. But it is convenient to deal with other relaxation criteria that do not introduce extra parameters since the controls may depend on them in a very sensitive way.

The two-grid method is a very natural way of introducing such relaxation. It is based on the idea of relaxing the final condition to avoid the divergence of controls due to the need of controlling high frequency spurious oscillations. To be more precise, the semi-discrete analogue of the exact controllability final condition (2.3) is

$$y_j(T) = z_j^0, \quad y_j'(T) = z_j^1, \qquad j = 0, \ldots, N+1. \tag{2.5}$$

But, as it is by now well-known (see [30]), under the final requirement (2.5), controls diverge as $h \to 0$ even for the linear wave equation.

In the two-grid algorithm, the final condition (2.5) is relaxed to

$$\Pi_h\big(Y(T)\big) = \Pi_h(Z^0), \qquad \Pi_h\big(Y'(T)\big) = \Pi_h(Z^1), \tag{2.6}$$

where $Y(t)$ and $Y'(t)$ stand for the vector-valued unknowns

$$Y(t) = \big(y_0(t), \ldots, y_{N+1}(t)\big), \qquad Y'(t) = \big(y_0'(t), \ldots, y_{N+1}'(t)\big).$$

We shall also use the notation Y_h for Y when passing to the limit to better underline the dependence on the parameter h. Π_h is the projection operator so that

$$\Pi_h(G) = \left(\frac{1}{2}\left(g_{2j+1} + \frac{1}{2}g_{2j} + \frac{1}{2}g_{2j+2}\right)\right)_{j=0,\dots,\frac{N+1}{2}-1}, \tag{2.7}$$

with $G = (g_0, g_1, \dots, g_N, g_{N+1})$. Note that the projection $\Pi_h(G)$ is a vector of dimension $(N+1)/2$. Thus, roughly speaking, the relaxed final requirement (2.6) only guarantees that half of the state of the numerical scheme is controlled. Despite this fact, the formal limit of (2.6) as $h \to 0$ is still the exact controllability condition (2.3) on the continuous wave equation.

The main result of this section is as follows:

Theorem 2.1. *Assume that the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is such that*

$$f \text{ is globally Lipschitz.} \tag{2.8}$$

Let $T_0 > 0$ be such that the two-grid algorithm for the control of the linear wave equation converges for all $T > T_0$.

Then, the algorithm converges for the semilinear system (2.1) *too for all $T > T_0$. More precisely, for all $(y^0, y^1) \in H^s(0, 1) \times H^{s-1}(0, 1)$ with $s > 0$, there exists a family of controls $\{v_h\}_{h>0}$ for the semi-discrete system* (2.4) *such that the solutions of* (2.4) *satisfy the relaxed controllability condition* (2.6) *and*

$$v_h(t) \to v(t) \qquad \text{in } L^2(0, T),\ h \to 0 \tag{2.9}$$

$$(Y_h, Y_h') \to (y, y_t) \quad \text{in } L^2(0, T; L^2(0, 1) \times H^{-1}(0, 1)) \tag{2.10}$$

where y is the solution of the semilinear wave equation (2.1) *and v is a control such that the state y satisfies the final requirement* (2.3).

Remark 2.2. Several remarks are in order.

• The controllability of the semilinear wave equation (2.1) under the globally Lipschitz assumption (2.8) on the nonlinearity was proved in [25] in $1-d$ and in the multi-dimensional case. The proof of Theorem 2.1 is based on an adaptation of the arguments in [25] to the two-grid approximation scheme.

Whether the two-grid algorithm applies under the weaker and sharp growth condition (2.2) is an open problem. The difficulty for doing that is that the two existing proofs allowing to deal with the semilinear wave equation under the weaker growth condition (2.2) are based, on a way or another, on the sidewise solvability of the wave equation, a property that the semi-discrete scheme fails to have.

• Theorem 2.1 holds for a sufficiently large time T_0. The requirement on T_0 is that, in the linear case ($f \equiv 0$), the two-grid algorithm converges for all $T > T_0$. This was proved to hold for $T > 4$ in [17]. The proof in [17] is based on the obtention of the corresponding observability inequality for the solutions of the adjoint semi-discrete wave equation by multiplier techniques. Later on this result was improved in [16] using a variant of the classical Ingham inequality obtaining the sharp minimal control time $T_0 = 2\sqrt{2}$.

Note that the minimal time for controllability of the continuous wave equation (2.1) is $T = 2$.[1] However this minimal time may not be achieved by the two-grid algorithm as described here since, despite it filters the spurious high frequency numerical solutions, it is compatible with the existence of wave packets travelling with velocity smaller than 1, and this excludes the controllability in the minimal time $T = 2$. The two-grid algorithm can be further improved to get smaller minimal times by considering other projection operators Π_h, obtained by means of the two-grid approach we shall describe below but with ratio $1/2^{\ell}$, for some $\ell \geq 2$, instead of the ratio $1/2$. This idea has been used successfully in [7] when proving dispersive estimates for conservative semi-discrete approximation schemes of the Schrödinger equation. When diminishing the ratio between grids, the filtering that the two-grid algorithm introduces concentrates the solutions of the numerical problem on lower and lower frequencies for which the velocity of propagation becomes closer and closer to that of the continuous wave equation. In that way the minimal controllability time may be made arbitrarily close to that of the wave equation $T = 2$ by means of the two-grid approach.

• In the statement of Theorem 2.1 we have chosen initial data for (2.1) in the space $(y^0, y^1) \in H^s(0, 1) \times H^{s-1}(0, 1)$, but we have not explained how the initial data for the semi-discrete system (2.4) have to be taken. The simplest way for doing that is taking as initial data for (2.4) the truncated Fourier series of the continuous initial data (y^0, y^1), involving only the first N Fourier modes. One can also define the discrete initial data by taking averages of the continuous ones on the intervals $[x_j - h/2, x_j + h/2]$ around the mesh-points.

• The meaning of the convergence property (2.10) needs also to be made precise. This may be done by extending the semi-discrete state $(Y_h(t), Y_h'(t))$ into a continuous one $(y_h(x, t), y_h'(x, t))$ and then proving convergence (2.10) for the extended one. This extension may be defined at least in two different ways. Either by extending the Fourier representation of Y_h or rather by using a standard piecewise linear and continuous extension. We refer to [28] and [19] where these two extensions have been used in similar limit processes.

• In the statement of Theorem 2.1 the initial data are assumed to be in $H^s(0, 1) \times H^{-1+s}(0, 1)$ for some $s > 0$, which is a slightly stronger regularity assumption than the one needed for the semilinear wave equation (2.1) to be controllable ($L^2(0, 1) \times H^{-1}(0, 1)$). This is probably a purely technical assumption but it is needed for the method we develop here to apply. The same difficulty arises in the context of the continuous semilinear wave equation [25]. This extra regularity condition for the continuous wave equation was avoided in [1] and [26] but using the very special property of the $1-d$ wave equation of being well-posed in the sidewise sense.

[1] By minimimal control time we mean that the controllability property holds for all time T which is greater than 2. Thus, this does not necessarily mean that controllability occurs for time $T = 2$.

In the context of the problem of numerical approximation we are working this difficulty seems hard to avoid even at the level of passing to the limit as $h \to 0$ on the state equations. Indeed, this requires passing to the limit, in particular, on the nonlinear terms and this seems hard to achieve in the $L^2(0,\ 1) \times H^{-1}(0,\ 1)$-setting because the corresponding states Y_h would then be merely bounded in $C([0,\ T];\ L^2(0,\ 1)) \cap C^1([0,\ T];\ H^{-1}(0,\ 1))$, which seems to be insufficient to guarantee compactness and the convergence of the nonlinear term.

Proof of Theorem 2.1. To simplify the presentation we assume that the final target is the null trivial state $z^0 \equiv z^1 \equiv 0$, although the same proof applies in the general case. We proceed in several steps.

Step 1. *Two-grid controllability of the semi-discrete system* (2.4). First of all, following the standard fixed point argument ([25]), we prove that, for $h > 0$ fixed, the semilinear system (2.4) is controllable. In fact this argument allows proving that (2.4) is exactly controllable for all $T > 0$. But, as we mentioned above (see [30]), the controls fail to be bounded as $h \to 0$. It is precisely to guarantee that the controls are bounded that we need to relax the final condition to the weaker two-grid one (2.6) and the time T is needed to be large enough as in the statement of Theorem 2.1.

To simplify the presentation we assume that $f \in C^1(\mathbb{R};\ \mathbb{R})$ and $f(0) = 0$, although the proof can be easily adapted to globally Lipschitz nonlinearities. We then introduce the continuous function

$$g(z) = \begin{cases} f(z)/z, & z \neq 0, \\ f'(0), & z = 0. \end{cases} \tag{2.11}$$

Given any semi-discrete function $Z = Z(t) \in C([0,\ T];\ \mathbb{R}^{N+2})$ we consider the linearized wave equation

$$\begin{cases} y_j'' + \frac{2y_j - y_{j+1} - y_{j-1}}{h^2} + g(z_j)y_j = 0, & j = 1, \dots, N;\ 0 < t < T, \\ y_0(t) = y_j^0,\ y_{N+1}(t) = v(t), & 0 < t < T, \\ y_j(0) = y_j'(0) = y_j^1, & j = 0, \dots, N+1. \end{cases} \tag{2.12}$$

We proceed by a classical fixed point argument (see [25]). This requires essentially proving that:

a) For all $Z = Z(t)$ as above (2.12) is two-grid controllable in the sense of (2.6);

b) The mapping $\mathcal{N}(Z) = Y$ has a fixed point.

To be more precise, we shall identify uniquely a control of minimal $L^2(0,\ T)$-norm v (which, obviously, depends on Z. Thus, in some cases we shall also denote it as v_Z). In this way the controlled trajectory $Y = Y_Z$ will also be uniquely determined and the nonlinear map $\mathcal{N}$ well defined. The problem is then reduced to proving that the map $\mathcal{N}$ has a fixed point. Indeed, if $Z = Y$, and, consequently, $g(z_j)y_j = f(y_j)$ for all $j = 1, \ldots, N$, then Y is also solution of the semilinear semi-discrete equation (2.1) and, of course, satisfies the two-grid relaxed final requirement (2.6).

The existence of the fixed point of $\mathcal{N}$ is consequence of Schauder's fixed point Theorem. The key point to apply it is to show a bound on the two-grid control for the linearized equation (2.12) which is independent of Z, i.e. the existence of $C > 0$ such that

$$\|v_Z\|_{L^2(0,T)} \leq C \quad \text{for all } Z \in C([0, T]; \mathbb{R}^{N+2}). \tag{2.13}$$

Here and in the sequel we denote by v_Z the control of the linearized system (2.12) to underline the fact that the control depends on the potential $g(Z)$ and thus on Z.

To do that we argue as in [17], reducing the problem to the obtention of a suitable observability inequality for the adjoint system:

$$\begin{cases} \varphi_j'' + \frac{2\varphi_j - \varphi_{j+1} - \varphi_{j-1}}{h^2} + g(z_j)\varphi_j = 0, & j = 1, \ldots, N, \ 0 < t < T, \\ \varphi_0 = \varphi_{N+1} = 0, & 0 < t < T, \\ \varphi_j(T) = \varphi_j^0, \ \varphi_j'(T) = \varphi_j^1, & j = 1, \ldots, N. \end{cases} \tag{2.14}$$

For doing that, however, system (2.14) has to be considered only in the class of slowly oscillating data obtained as extensions to the fine grid (the original one, of size h) of data defined on a coarse grid of size $2h$. In other words, we consider the class of data

$$\mathcal{V}_h = \left\{ \Phi = (\varphi_0, \ldots, \varphi_{N+1}) : \varphi_{2j+1} = \frac{\varphi_{2j} + \varphi_{2j+2}}{2}, \ j = 0, \ldots, \frac{N-1}{2} \right\}. \tag{2.15}$$

Note that any vector in $\mathcal{V}_h$ is completely determined by its values on the grid of mesh-size $2h$. Implicitly we are assuming that $1/2h$ is an integer number so that $(N-1)/2 = 1/2h - 1$ is an integer too.

In [17] and [16] it was proved that for $T > T_0$, where T_0 is as in Remark 2.2, the following observability inequality holds:

$$E_0 \leq C \int_0^T \left| \frac{\varphi_N}{h} \right|^2 dt, \tag{2.16}$$

with $C > 0$ independent of $h > 0$ and for all solution $\Phi = (\varphi_0, \ldots, \varphi_{N+1})$ of (2.14) with data $(\Phi^0, \Phi^1) \in \mathcal{V}_h \times \mathcal{V}_h$ when $g \equiv 0$. Here E_0 stands for the total energy of solutions at time $t = T$, which is constant in time when $g \equiv 0$:

$$E(t) = \frac{h}{2} \sum_{j=0}^{N} \left[|\varphi_j'|^2 + \left| \frac{\varphi_{j+1} - \varphi_j}{h} \right|^2 \right]. \tag{2.17}$$

At this point it is important to emphasize that the key ingredient of the proof of convergence for the two-grid algorithm for the linear wave equation is precisely that the observability constant C in (2.16) is uniform, independent of h.

Let us now address the perturbed problem (2.14). We first observe that, because of the globally Lipschitz assumption on f, the function g is uniformly bounded, i.e.

$$\|g\|_{L^\infty(\mathbb{R})} \leq L, \tag{2.18}$$

L being the Lipschitz constant of f. Therefore for all $Z \in C([0, T]; \mathbb{R}^{N+2})$ it follows that

$$\|g(Z)\|_{L^\infty(0, T; \mathbb{R}^{N+2})} \leq L. \tag{2.19}$$

System (2.14) can then be viewed as a family of perturbed semi-discrete wave equations, the perturbations with respect to the conservative wave equation being a family of zero order bounded potentials. Then a standard perturbation argument allows showing that (2.16) holds for system (2.14) too, with, possibly, a larger observability constant, depending on L, but independent of Z. In fact, arguing by contradiction, since $h > 0$ is fixed and we are therefore dealing with finite-dimensional dynamical systems, the problem is reduced to show that the following unique continuation property holds for all Z:

$$\text{If } \varphi_N(t) = 0,\ 0 < t < T, \text{ then } \Phi \equiv 0. \tag{2.20}$$

This property is easy to prove by induction. Indeed, using the boundary condition $\varphi_{N+1} \equiv 0$ and the fact that $\varphi_N \equiv 0$, and writing the equation (2.14) for $j = N$ we deduce that

$$\frac{\varphi_{N-1}}{h^2} = \varphi_N'' + \frac{2\varphi_N - \varphi_{N-1}}{h^2} + g(z_N)\varphi_N = 0. \tag{2.21}$$

This implies that $\varphi_{N-1} = 0$. Repeating this argument we deduce that $\Phi \equiv 0$.

Once (2.16) holds for the solutions of (2.14) with initial data in $\mathcal{V}_h \times \mathcal{V}_h$, uniformly on Z, system (2.14) turns out to be controllable in the sense of (2.6) with an uniform bound on the control, independent of Z, i.e.

$$\|v\|_{L^2(0,T)} \leq C(h, \|(Y^0, Y^1)\|, L, T) \quad \text{for all } Z. \tag{2.22}$$

We emphasize that the bound (2.22), in principle, depends on the time of control T, the mesh-size h, the Lipschitz constant L of the nonlinearity f and the norm of the initial data to be controlled, but is independent of Z.

As a consequence of (2.22) a similar estimate can be obtained for the state Y solution of (2.12), i.e.

$$\|Y\|_{C^1([0,T]; \mathbb{R}^{N+2})} \leq C \quad \text{for all } Z. \tag{2.23}$$

This allows applying the Schauder's fixed point theorem to the map $\mathcal{N}$, which turns out also to be compact from $L^2(0, T; \mathbb{R}^{N+2})$ into itself, thanks to (2.23). In this way we conclude that, for all $h > 0$, system (2.4) is controllable in the sense of (2.6).

Step 2. *Uniform controllability with respect to* h. In the previous step we have proved the controllability of (2.4) but with estimates on controls and states depending on h.

In order to pass to the limit as $h \to 0$ we need to get a bound on controls and states which is independent of h. For doing that we need to assume that $T > T_0$ (so that the linear unperturbed numerical schemes are uniformly, with respect to $h > 0$, two-grid controllable) and that the initial data (y^0, y^1) belong to $H^s(0, 1) \times H^{s-1}(0, 1)$.

The last requirement is important to get the compactness of the nonlinear term. Indeed, in that setting the control for the continuous wave equation (2.1) belongs to $H^s(0, T)$ rather than $L^2(0, T)$ and the controlled trajectory y then belongs to $C([0, T]; H^s(0, 1)) \cap C^1([0, T]; H^{s-1}(0, 1))$. This guarantees the required compactness properties to deal with the nonlinear term $f(y)$ in (2.1). Indeed, when passing to the limit, the pointwise convergence of the state in $(0, 1) \times (0, T)$ is needed and this is achieved by means of the extra H^s regularity imposed on the initial data (see [25]). This is necessary both when treating the continuous equation (2.1) by fixed point arguments and also when dealing with numerical approximation issues and limit processes as $h \to 0$.

To analyze the controllability of the systems under consideration in $H^s(0, 1) \times H^{s-1}(0, 1)$, we first need to analyze the H^{-s}-version of the observability inequality (2.16), namely:

$$E_{0,-s} \le C_s \left\| \frac{\varphi_N}{h} \right\|^2_{H^{-s}(0,T)}. \tag{2.24}$$

Inequality (2.24) may be proved for the adjoint system (2.14) in the absence of the potential induced by the nonlinearity, i.e. for

$$\begin{cases} \psi_j'' + \frac{2\psi_j - \psi_{j+1} - \psi_{j-1}}{h^2} = 0, & j = 1, \dots, N, \ 0 < t < T, \\ \psi_0 = \psi_{N+1} = 0, & 0 < t < T, \\ \psi_j(T) = \psi_j^0, \ \psi_j(T) = \psi_j^1, & j = 1, \dots, N. \end{cases} \tag{2.25}$$

More precisely, for $0 < s < 1/2$ and $T > T_0$ as in Theorem 2.1, there exists a constant C_s such that (2.24) holds for all solution ψ of (2.25) with initial data in $\mathcal{V}_h \times \mathcal{V}_h$ and all $h > 0$. We emphasize that the constant C_s is independent of h.

In (2.24) $E_{0,-s}$ stands for the H^{-s} version of the energy of system (2.25), which is constant in time. It can be defined easily by means of the Fourier expansion of solutions and it is then the discrete analogue of the continuous energy

$$E_{0,-s} = \frac{1}{2}\left[\|\psi^0\|^2_{H^{1-s}(0,1)} + \|\psi^1\|^2_{H^{-s}(0,1)} \right] \tag{2.26}$$

which is constant in time for the solutions of the unperturbed adjoint wave equation

$$\begin{cases} \psi'' - \psi_{xx} = 0, & 0 < x < 1, 0 < t < T, \\ \psi(0, t) = \psi(1, t) = 0, & 0 < t < T, \\ \psi(x, T) = \psi^0(x), \ \psi_t(x, T) = \psi^1(x), & 0 < x < 1. \end{cases} \tag{2.27}$$

The inequality (2.24) may be obtained, as (2.16), by the two methods mentioned above:

- It can be proved as a consequence of (2.16) directly using interpolation arguments (see, for instance, [25]);
- It can also be obtained by the variant of the Ingham inequality in [16].

Once (2.24) is proved for the unperturbed system (2.25), uniformly on $h > 0$, we are in conditions to prove it for the perturbed system (2.14) uniformly on $h > 0$ and Z too. To do it we use a classical perturbation and compactness argument (see [25]).

We decompose the solution Φ of (2.14) as $\Phi = \Psi + \Sigma$ where Ψ solves the unperturbed system (2.25) with the same data $(\Phi^0,\ \Phi^1)$ as Φ itself and where the remainder $\Sigma = (\sigma_0, \dots, \sigma_{N+1})$ solves

$$\begin{cases} \sigma_j'' + \frac{2\sigma_j - \sigma_{j+1} - \sigma_{j-1}}{h^2} = -g(z_j)\varphi_j, & j = 1, \dots, N,\ 0 < t < T, \\ \sigma_0 = \sigma_{N+1} = 0, & 0 < t < T, \\ \sigma_j(T) = \sigma_j'(T) = 0, & j = 1, \dots, N. \end{cases} \tag{2.28}$$

In view of (2.24), which is valid for Ψ, we deduce that

$$E_{0,-s} \le 2C_s \Big[\Big\| \frac{\varphi_N}{h} \Big\|^2_{H^{-s}(0,T)} + \Big\| \frac{\sigma_N}{h} \Big\|^2_{H^{-s}(0,T)} \Big]. \tag{2.29}$$

Using discrete multipliers (see [8]) it follows that

$$\Big\| \frac{\sigma_N}{h} \Big\|_{L^2(0,T)} \le C \|\{g(z_j)\varphi_j\}\|_{L^2(0,T;\,\ell_h^2)} \tag{2.30}$$

with a constant C which depends on T but is independent of h.

In (2.30) we use the notation

$$\|\{p_j\}\|_{L^2(0,T;\,\ell_h^2)} = \Big[h \int_0^T \sum_{j=1}^N p_j^2(t)\, dt \Big]^{1/2} \tag{2.31}$$

which is simply the L^2-norm, scaled to the mesh-size $h > 0$.

Combining (2.30)–(2.31) and using that the nonlinearity g is uniformly bounded we deduce that

$$E_{0,-s} \le C \Big[\Big\| \frac{\varphi_N}{h} \Big\|^2_{H^{-s}(0,T)} + \|\Phi\|^2_{L^2(0,T;\,\ell_h^2)} \Big], \tag{2.32}$$

for every solution Φ of (2.14) with data $(\Phi^0,\ \Phi^1)$ in $\mathcal{V}_h \times \mathcal{V}_h$, every $h > 0$ and Z.

To conclude we can apply a compactness-uniqueness argument whose details may be found in [28] where it was fully developed in the context of the $2-d$ semi-discrete wave equation. It consists simply in showing, by contradiction, that there exists an uniform constant $C > 0$ such that

$$\|\Phi\|_{L^2(0,T;\,\ell_h^2)} \le C \Big\| \frac{\varphi_N}{h} \Big\|_{H^{-s}(0,T)} \tag{2.33}$$

for every solution Φ of (2.14), every $h > 0$ and Z. To do it we assume that there exists a sequence $h \to 0$, potentials of the form $g(Z_h)$ and initial data in $\mathcal{V}_h \times \mathcal{V}_h$ for which (2.33) fails and, consequently,

$$\left\| \frac{\varphi_N}{h} \right\|_{H^{-s}(0,\,T)} \to 0, \quad h \to 0 \tag{2.34}$$

$$\|\Phi\|_{L^2(0,\,T;\,\ell_h^2)} = 1. \tag{2.35}$$

Combining (2.32), (2.34) and (2.35), the corresponding sequence of data $(\Phi_h^0,\ \Phi_h^1)$ turns out to be bounded in $H^{1-s}(0, 1) \times H^{-s}(0, 1)$ (at this point we are implicitly working with the piecewise linear extension of the data). By the well-posedness of (2.14) in these spaces the corresponding solutions Φ_h turn out to be bounded in $L^\infty(0,\ T;\ H^{s-1}(0, 1)) \cap W^{1,\infty}(0,\ T;\ H^{-s}(0, 1))$. Therefore, they are relatively compact in $L^2(0,\ T;\ L^2(0, 1))$. Passing to the limit as $h \to 0$ we obtain a solution φ of an adjoint wave equation of the form

$$\begin{cases} \varphi'' - \varphi_{xx} + a(x,\ t)\varphi = 0, & 0 < x < 1,\ 0 < t < T, \\ \varphi(0,\ t) = \varphi(1,\ t) = 0, & 0 < t < T, \\ \varphi(x,\ T) = \varphi^0(x),\ \varphi_t(x,\ 0) = \varphi^1(x), & 0 < x < 1, \end{cases} \tag{2.36}$$

such that

$$\partial_x \varphi(1,\ t) = 0, \quad 0 < t < T \tag{2.37}$$

and

$$\|\varphi\|_{L^2(0,\,T;\,L^2(0,\,1))} = 1. \tag{2.38}$$

This is clearly a contradiction since, in view of the unique continuation property of the solutions of the wave equation (2.36), (2.37) implies that $\varphi \equiv 0$, which is incompatible with (2.38). This is true because $T > T_0$ and, in particular $T > 2$.

In this argument the bounded limit potential $a = a(x,\ t)$ in (2.36) arises as weak-$*$ limit of the discrete ones $g\ (Z_h)$, (of its piecewise linear extension to $0 < x < 1,\ 0 < t < T$, to be more precise). Therefore a also fulfills the bound $\|a\|_\infty \leq L$, L being the Lipschitz constant of f.

For this argument to apply one needs to pass to the limit in the potential perturbation $g\ (Z_h)\ \Phi_h$ in (2.14). This can be done because of the strong convergence of Φ_h (of its extension to $0 < x < 1$) in $L^2((0,\ 1) \times (0,\ T))$.

Once (2.24) is known to hold for all $h > 0$ and all data in $\mathcal{V}_h \times \mathcal{V}_h$ this allows proving the uniform controllability of (2.4) in the spaces $H^s(0, 1) \times H^{-1+s}(0, 1)$, in the two-grid sense (2.6). This can be done applying the fixed point argument in Step 1. More precisely, it follows that there exists a family of controls $v_h \in H^s(0,\ T)$, with an uniform bound

$$\|v_h\|_{H^s(0,\,T)} \leq C \left\| \left(Y_h^0,\ Y_h^1\right) \right\|_{H^s(0,1)\times H^{s-1}(0,1)} \tag{2.39}$$

such that the solutions Y_h of (2.4) satisfy (2.6).

Step 3. *Two-grid observability* $\Longrightarrow$ *Two-grid controllability.* For the sake of completeness, let us show how the two-grid control of (2.12) can be obtained as a consequence of the observability inequality (2.24) for the solutions of the adjoint wave equation (2.14) with initial data in the class $\mathcal{V}_h \times \mathcal{V}_h$ in (2.15) of slowly oscillating data.

We introduce the functional

$$J_h\left(\Phi^0,\ \Phi^1\right) = \frac{1}{2}\left\|\frac{\varphi_N}{h}\right\|^2_{H^{-s}(0,\,T)} + h\sum_{j=1}^{N}\left[y_j^0\varphi_j'(0) - y_j^1\varphi_j(0)\right], \tag{2.40}$$

which is continuous and convex. Moreover, in view of (2.24), the functional $J_h\colon \mathcal{V}_h \times \mathcal{V}_h \to \mathbb{R}$ is uniformly coercive. Let us denote by $(\Phi_h^{0,*},\ \Phi_h^{1,*})$ the minimizer of J_h over $\mathcal{V}_h \times \mathcal{V}_h$. Then,

$$\left\langle DJ_h\left(\Phi_h^{0,*},\ \Phi_h^{1,*}\right),\ \left(\Phi^0,\ \Phi^1\right)\right\rangle = 0 \tag{2.41}$$

for all $(\Phi^0,\ \Phi^1) \in \mathcal{V}_h \times \mathcal{V}_h$. This implies that

$$\left(\frac{\varphi_N^*}{h},\ \frac{\varphi_N}{h}\right)_{H^{-s}(0,\,T)} + h\sum_{j=1}^{N}\left[y_j^0\varphi_j'(0) - y_j^1\varphi_j(0)\right] = 0 \tag{2.42}$$

where Φ_h^* stands for the solution of (2.14) with the minimizer $(\Phi_h^{0,*},\ \Phi_h^{1,*})$ as data and Φ the solution with data $(\Phi^0,\ \Phi^1)$.

We now choose the control

$$v_h = I_s\frac{\varphi_N^*}{h}, \tag{2.43}$$

where $I_s\colon H^{-s}(0,\ T) \to H^s(0,\ T)$ is the canonical duality isomorphism.

Equation (2.42) then reads

$$\int_0^T v_h\frac{\varphi_N}{h}dt + h\sum_{j=0}^{N}\left[y_j^0\varphi_j'(0) - y_j^1\varphi_j(0)\right] = 0. \tag{2.44}$$

On the other hand, using (2.43) as control in (2.12), multiplying by Φ the solution of the adjoint system (2.14), adding on $j = 1, \dots, N$ and integrating by parts with respect to $t \in (0,\ T)$, we deduce that

$$\int_0^T v\frac{\varphi_N}{h}dt + h\sum_{j=1}^{N}\left[y_j^0\varphi_j'(0) - y_j^1\varphi_j(0)\right] - h\sum_{j=1}^{N}\left[y_j(T)\varphi_j^1 - y_j'(T)\varphi_j^0\right] = 0. \tag{2.45}$$

Combining (2.44)–(2.45) we deduce that the solution Y_h of (2.12) satisfies

$$h\sum_{j=1}^{N}\left[y_j(T)\varphi_j^1 - y_j'(T)\varphi_j^0\right] = 0 \quad \text{for all } (\varphi^0,\ \varphi^1) \in \mathcal{V}_h \times \mathcal{V}_h. \tag{2.46}$$

This means that both $Y_h(T)$ and $Y_h'(T)$ are perpendicular to $\mathcal{V}_h$. This is equivalent to the two-grid control requirement (2.6) with $z^0 \equiv z^1 \equiv 0$.

In view of this construction and using the observability inequality (2.24), which is uniform with respect to $h > 0$ and Z, we can obtain uniform bounds on the controls. Indeed, by (2.43) we have

$$\|v_h\|_{H^s(0,\,T)} = \left\| \frac{\varphi_N^*}{h} \right\|_{H^{-s}(0,\,T)}. \tag{2.47}$$

On the other hand, the minimizer $(\Phi_h^{0,\,*},\ \Phi_h^{1,\,*})$ is such that

$$J\big(\Phi_h^{0,\,*},\ \Phi_h^{1,\,*}\big) \le 0 \tag{2.48}$$

and this implies

$$\begin{aligned} \frac{1}{2}\|\frac{\varphi_N^*}{h}\|^2_{H^{-s}(0,\,T)} &\le \Big| h \sum_{j=1}^{N} \big[y_j^0 \varphi_j^{*'}(0) - y_j^1 \varphi_j^*(0) \big] \Big| \\ &\le \|(Y^0,\ Y^1)\|_{H^s(0,1)\times H^{s-1}(0,1)} \sqrt{E^*_{0,\,-s}} \end{aligned} \tag{2.49}$$

where $E^*_{0,\,-s}$ denotes the $E_{0,\,-s}$ energy of the minimizer $(\Phi_h^{0,\,*},\ \Phi_h^{1,\,*})$.

Combining (2.48), (2.49) and the observability inequality (2.24) we deduce that

$$\|v_h\|_{H^s(0,\,T)} \le 2\sqrt{C_s}\|(Y^0,\ Y^1)\|_{H^s(0,1)\times H^{s-1}(0,1)}, \tag{2.50}$$

where C_s is the same constant as in (2.24). In particular, the bound (2.50) on the control is independent of $h > 0$ and Z.

Step 4. *Passing to the limit.* Using the uniform bound (2.39) it is easy to pass to the limit and get the null-controllability of the semilinear wave equation (2.1).

Indeed, as a consequence of (2.39) and by the well-posedness of (2.4) we deduce that the controlled state Y_h is uniformly bounded in $L^\infty(0,\,T;\,H^s(0,1)) \cap W^{1,\,\infty}(0,\ T;\ H^{s-1}(0,1))$.

By extracting subsequences we have

$$v_h \rightharpoonup v \text{ weakly in } H^s(0,\ T) \tag{2.51}$$

$$Y_h \rightharpoonup y \text{ weakly in } L^2(0,\ T;\ H^s(0,\ 1)) \cap H^1(0,\ T;\ H^{s-1}(0,\ 1)). \tag{2.52}$$

Consequently, in particular,

$$v_h \to v \text{ strongly in } L^2(0,\ T) \tag{2.53}$$

$$Y_h \to y \text{ strongly in } L^2((0,\ 1)\times(0,\ T)). \tag{2.54}$$

These convergences suffice to pass to the limit in (2.4) and to get (2.1). The strong convergence (2.54) is particularly relevant when doing that since it allows passing to the limit in the nonlinearity.

Here the convergence of the states Y_h may be understood in the sense that its extensions to functions defined for all $0 < x < 1$ converge.

One can also check that the limit state $y = y(x, t)$ satisfies the final exact controllability requirement (2.3) as a consequence of the two-grid relaxed version (2.6) that the semi-discrete state Y_h satisfies. This can be done either by transposition or by compactness.

This concludes the sketch of the proof of Theorem 2.1.

Remark 2.3. Several remarks are in order:

- The proof we have given can be adapted to other equations and schemes. In particular it applies to the two-grid finite element approximation of (2.1).
- In [2] a mixed finite-element discretization scheme has been introduced for which the uniform controllability property holds without requiring any filtering or two-grid adaptation. The arguments we have developed here can also be adapted to prove convergence of that method in the semilinear case under the globally Lipschitz assumption on the nonlinearity f.
- In [12] it was proved that the standard finite-difference semi-discretization for the exact controllability of the following beam equation converges without filtering or two-grid adaptation:

$$y_{tt} + y_{xxxx} = 0.$$

  The method of proof of Theorem 2.1 allows showing that the same is true in the semilinear context too.
- The proof of convergence of the two-grid control algorithm is still to be developed for numerical approximations of the wave equation in the multi-dimensional case. But, in view of the proof of Theorem 2.1, which can be easily adapted to the multi-dimensional framework, we can say that, if convergence is proved in the linear case, the same will hold in the semilinear one too, for globally Lipschitz nonlinearities.
- For the semilinear wave equation (2.1) the *local* null-controllability can be proved in wider classes of nonlinearities satisfying $f'(0) = 0$. Here by local null-controllability we refer to the property that sufficiently small initial data can be driven to the null state, i.e. to the existence of $\delta > 0$ such that the control driving the solution to the final equilibrium $\{0, 0\}$ exists for all initial data $\{y^0, y^1\}$ such that

$$\|y^0\|_{L^2(0,1)} + \|y^1\|_{H^{-1}(0,1)} \le \delta.$$

  It can be proved as a consequence of the controllability of the linear wave equation applying the inverse function theorem around the null state. In order to guarantee the well-posedness of the semilinear wave equation (2.1) in

$L^2(0, 1) \times H^{-1}(0, 1)$ one also needs to impose a growth condition on the nonlinearity of the form

$$|f'(s)| \leq C|s| \quad \text{for all } s \in \mathbb{R}.$$

But this allows proving local controllability for quadratic nonlinearities, for instance (see [24]).

The method of proof of Theorem 2.1 can be used to prove the convergence of the two-grid method in that context of local controllability too.

- In [24] it was also observed that for nonlinearities with the good sign property, for instance, for

  $$f(s) = |s|^{p-1}s \quad \text{for all } s \in \mathbb{R}$$

  with $1 < p \leq 2$, every initial datum may be driven to zero for a sufficiently large time. For doing that one first uses the exponential decay of solutions with boundary feedback ([9]) to later apply the local controllability property when the solution becomes small enough.

  To adapt that result to the framework of the two-grid scheme one could need a uniform (with respect to h) stabilization result for the numerical schemes with boundary feedback. However it is well known that, due to the lack of uniform boundary observability as $h \to 0$, the uniform stabilization property fails. In [19] and [20] (see also [18]) the uniform (with respect to h) exponential decay property was proved but by adding a viscous damping term distributed all along the mesh. In view of the efficiency of the two-grid approach at the level of controllability, one would expect the uniform (with respect to $h > 0$) exponential decay property to hold for initial data in the space $\mathcal{V}_h \times \mathcal{V}_h$ without the extra viscous damping term. But this property does not seem to happen. Indeed, when trying to obtain the uniform exponential decay from the uniform observability inequality a technical difficulty appears since the space $\mathcal{V}_h \times \mathcal{V}_h$ is not invariant under the flow of the semi-discrete wave equation. Thus, the observability inequality we have obtained in the time interval $[0, T]$ with $T > T_0$ can not be extended for all $t \geq 0$, a fact that would be needed for proving the exponential decay. But in fact, the situation is much worse and the uniform exponential decay fails to hold since, despite of the fact that the two-grid initial data have a distribution of energy so that most of it is concentrated on the low frequencies, as time evolves, this partition of energy is lost because, precisely, high frequency components are weakly dissipated. The apparently purely technical difficulty for proving the uniform decay turns out to be in fact the reason for the lack of uniform decay.

- Recently it has also been proved that $1 - d$ semilinear wave equations are controllable in the sense that two different equilibria can be connected by a controlled trajectory provided they belong to the same connected component of the set of stationary solutions (see [3]). This holds without any sign restriction on the nonlinearity and therefore without excluding blow-up phenomena to occur. Proving the convergence of the two-grid algorithm in what concerns that result is an open problem too.

- The main drawback of the arguments we have used in the proof of Theorem 2.1 is that they do not provide any explicit estimate on the cost of controlling the system with respect to the Lipschitz constant L of the nonlinearity. This is due to the use of compactness-uniqueness arguments in the obtention of the uniform (with respect to h) observability estimates (2.24) in the class of initial data $\mathcal{V}_h \times \mathcal{V}_h$. Therefore we may not recover by this method the property of controllability of the semilinear wave equation under the sharp superlinear growth condition (2.2).
- The existence of the convergent controls of the semi-discrete semilinear system has been proved by means of a fixed point method. This adds extra technical difficulties for its efficient computation. The most common tool to deal with such problem, for h fixed and after a suitable time discretization, is the Newton method with variable step. It is applicable in the present situation since the nonlinear map $\mathcal{N}$ under consideration is differentiable when the nonlinearity f in the equation is C^1. In each iteration of the Newton method, one is lead to solve a linearized control problem. But this one is solvable by means of a standard conjugate gradient algorithm ([6]) because of the uniform observability properties that are guaranteed to hold, as we have seen, due to the two-grid relaxation we have introduced. A complete numerical study of these issues is yet to be developed.
- The limit control v we have obtained can be proved to be a fixed point of the nonlinear map $\mathcal{N}$ that corresponds to the controllability of the semilinear continuous wave equation, based on the HUM controls of minimal $L^2(0, T)$-norm for the linearized wave equations. Thus, the controls we are dealing with both, for the continuous and the semi-discrete equation, belong to the same category.

3. Boundary control of the finite-difference space semi-discretizations of the heat equation

3.1. Problem formulation. This section is devoted to analyze the null controllability of the space semi-discretizations, by means of finite differences, of the heat equation in multi-dimensional domains. To simplify the presentation we focus on the $2-d$ case although the same results, with similar proofs, apply in any dimension $d \geq 2$.

The heat equation in bounded domains is known to be null-controllable from any open, non-empty subset of the domain or its boundary [5]. Thus, it is natural to analyze whether the control is the limit of the controls of the semi-discrete systems as the mesh-size tends to zero. But this turns out not to be the case even for the heat equation in the square when the control is applied on a strict subset of one of the segments constituting its boundary (see [30]).

In this section we prove a positive counterpart of that result. More precisely, we prove that convergence holds when the control acts on a whole side of the boundary. The proof uses the Fourier series development of solutions, which allows reducing the problem to a one-parameter family of controllable $1-d$ heat equations. As a consequence of that result we can prove convergence for general domains when the control is applied on the whole boundary. For that it is sufficient to extend the initial data in the original domain to data in a square containing it and then obtaining the controls on the boundary of the original domain as restrictions to the boundary of the states defined in the extended square.

The same results hold in any space dimension.

To be more precise, let Ω be the square $\Omega = (0,\pi)\times(0,\pi)$ of $\mathbb{R}^2$. Let Γ_0 be one side of its boundary, say $\Gamma_0 = \{(x_1, 0) : 0 < x_1 < \pi\}$.

Consider the heat equation with control on Γ_0:

$$\begin{cases} y_t - \Delta y = 0 & \text{in } \Omega \times (0,T), \\ y = 0 & \text{on } [\partial\Omega \setminus \Gamma_0] \times (0,T), \\ y = v & \text{on } \Gamma_0 \times (0,T), \\ y(x,0) = y^0(x) & \text{in } \Omega. \end{cases} \tag{3.1}$$

Here $y = y(x,t)$, with $x = (x_1, x_2)$, is the *state* and $v = v(x_1, t)$ is the *control*.

System (3.1) is well-known to be null-controllable in any time $T > 0$ (see Fursikov and Imanuvilov [5] and Lebeau and Robbiano [11]). More precisely, the following holds: *For any $T > 0$ and any $y^0 \in L^2(\Omega)$ there exists $v \in L^2(\Gamma_0 \times (0,T))$ such that the solution $y = y(x,t)$ of* (3.1) *satisfies*

$$y(x,T) \equiv 0. \tag{3.2}$$

Moreover, there exists a constant $C > 0$ depending on T but independent of the initial datum y^0 such that

$$\|v\|_{L^2(\Gamma_0\times(0,T))} \le C\|y^0\|_{L^2(\Omega)} \quad \text{for all } y^0 \in L^2(\Omega). \tag{3.3}$$

In fact the same result holds in a general smooth bounded domain Ω and with controls in any open non-empty subset Γ_0 of its boundary.

In the present setting, this result is equivalent to an observability inequality for the *adjoint heat equation*:

$$\begin{cases} \varphi_t + \Delta\varphi = 0 & \text{in } \Omega \times (0,T), \\ \varphi = 0 & \text{on } \partial\Omega \times (0,T), \\ \varphi(x,T) = \varphi^0(x) & \text{in } \Omega. \end{cases} \tag{3.4}$$

More precisely, it is equivalent to the existence of a positive constant $C > 0$ such that

$$\|\varphi(0)\|^2_{L^2(\Omega)} \le C \int_0^T \int_{\Gamma_0} \left|\frac{\partial\varphi}{\partial n}\right|^2 d\sigma dt \quad \text{for all } \varphi^0 \in L^2(\Omega). \tag{3.5}$$

Here and in the sequel by n we denote the unit exterior normal vector field and by $\partial\cdot/\partial n$ the normal derivative. In this case, over Γ_0, $\partial\cdot/\partial n = -\partial\cdot/\partial_{x_2}$.

Let us now consider the finite-difference space semi-discretizations of (3.1) and (3.4).

Given $N \in \mathbb{N}$ we set $h = \pi/(N+1)$ and we consider the mesh

$$x_{i,j} = (ih, jh), \quad i, j = 0, \ldots, N+1. \tag{3.6}$$

We now introduce the finite-difference semi-discretizations:

$$\begin{cases} y'_{j,k} + \frac{1}{h^2}(4y_{j,k} - y_{j+1,k} - y_{j-1,k} - y_{j,k+1} - y_{j,k-1}) = 0, \ (j,k) \in \Omega_h, \ 0 < t < T, \\ y_{j,k} = 0, \ (j,k) \in [\partial\Omega \setminus \Gamma_0]_h, \quad 0 < t < T, \\ y_{j,0} = v_j, \ j = 0, \ldots, N+1, \quad 0 < t < T, \\ y_{j,k}(0) = y^0_{j,k}, \quad (j,k) \in \Omega_h, \end{cases} \tag{3.7}$$

and

$$\begin{cases} \varphi'_{j,k} - \frac{1}{h^2}(4\varphi_{j,k} - \varphi_{j+1,k} - \varphi_{j-1,k} - \varphi_{j,k+1} - \varphi_{j,k-1}) = 0, \ (j,k) \in \Omega_h, \ 0 < t < T, \\ \varphi_{j,k} = 0, \quad (j,k) \in [\partial\Omega]_h, \ 0 < t < T, \\ \varphi_{j,k}(T) = \varphi^0_{j,k}, \quad (j,k) \in \Omega_h. \end{cases} \tag{3.8}$$

To simplify the notations, we have denoted by Ω_h (resp. $\partial\Omega_h$) the set of interior (resp. boundary) nodes, and by $[\partial\Omega \setminus \Gamma_0]_h$ the set of indices (j,k) so that the corresponding nodes belong to $\partial\Omega \setminus \Gamma_0$. Here and in the sequel $y_{j,k} = y_{j,k}(t)$ (resp. $\varphi_{j,k} = \varphi_{j,k}(t)$) stands for an approximation of the solution y of (3.1) (resp. φ of (3.4)) at the mesh-points $x_{i,j}$. On the other hand, v_j denotes the control that acts on the semi-discrete system (3.7) through the subset $[\Gamma_0]_h$ of the boundary. Note that the control does not depend of the index k since the subset of the boundary $[\Gamma_0]_h$ where the control is being applied corresponds to $k = 0$.

In order to simplify the notation we introduce the vector unknowns and control

$$Y_h = (y_{j,k})_{0\le j,k\le N+1}, \quad \Phi_h = (\phi_{j,k})_{0\le j,k\le N+1}, \quad V_h = (v_j)_{1\le j\le N}, \tag{3.9}$$

that we shall often denote simply by Y, Φ and V.

Accordingly, systems (3.7) and (3.8) read as follows:

$$\begin{cases} Y'_h + A_h Y_h = B_h V_h, \\ Y_h(0) = Y^0_h, \end{cases} \tag{3.10}$$

$$\begin{cases} \Phi_h' - A_h \Phi_h = 0, \\ \Phi_h(T) = \Phi_h^0. \end{cases} \tag{3.11}$$

We denote by A_h the usual positive-definite symmetric matrix associated with the five-point finite-different scheme we have employed in the discretization of the Laplacian so that

$$(A_h W)_{j,k} = \frac{1}{h^2}(4w_{j,k} - w_{j+1,k} - w_{j-1,k} - w_{j,k+1} - w_{j,k-1}), \tag{3.12}$$

for the inner nodes. In (3.11) the homogenous boundary conditions have been integrated by assuming simply that their values in the expression (3.12) have been replaced by the zero one. On the other hand the linear operator B_h in (3.10) is such that the action of the control v_j enters on those nodes which are neighbors to those of $[\Gamma_0]_h$, i.e. for $k = 1$, so that $[B_h V]_{j,k} = 0$ whenever $2 \leq k \leq N$ but $[B_h V]_{j,1} = -v_j/h^2$.

The null-controllability problem for system (3.10) reads as follows: Given $Y^0 \in \mathbb{R}^{N+2\times N+2}$ to find $V \in L^2(0, T; \mathbb{R}^N)$ such that the solution Y of (3.10) satisfies

$$Y(T) = 0. \tag{3.13}$$

On the other hand, the problem of observability for system (3.11) consists in proving the existence of $C > 0$ such that

$$\|\Phi(0)\|_h^2 \leq Ch \int_0^T \sum_{j=1}^{N} \Big|\frac{\phi_{j,1}}{h}\Big|^2 dt \tag{3.14}$$

for every solution Φ of (3.11).

In (3.14) $\|\cdot\|_h$ stands for the scaled Euclidean norm

$$\|\Phi\|_h = \Big[h^2 \sum_{j,k=0}^{N+1} |\phi_{j,k}|^2\Big]^{1/2} \tag{3.15}$$

and the right hand side term of inequality (3.14) represents the discrete version of the L^2-norm of the normal derivative in (3.5).

A similar problem can be formulated in general bounded smooth domains Ω. In that case, obviously, the domain Ω needs to be approximated by domains Ω_h whose boundaries are constituted by mesh-points. We first address the case of the square domain by Fourier series to later derive some consequences for general domains.

All this section is devoted to the problem of null control. Obviously the situation is different if the final requirement is relaxed to an approximate controllability condition. In that context, as a consequence of the null controllability of the limit heat equation and the convergence of the numerical algorithm it can be proved that the state Y_h at time $t = T$ can be driven to a final state of norm ε_h such that $\varepsilon_h \to 0$ as $h \to 0$. But, as mentioned above, this property fails in general in the framework of null controllability. At this point the work in [10] is also worth mentioning. There it was proved that, in the context of analytic semigroups, one can also get uniform bounds on the number of iterations needed for computing controls using conjugate gradient algorithms.

3.2. The square domain. The goal of this subsection is to prove that, as $h \to 0$, the controls V_h of (3.10) are uniformly bounded and converge in $L^2(\Gamma_0 \times (0,T))$ to the control of (3.1). All along this section we deal with controls of minimal L^2-norm, the so-called HUM controls.

In order to make this convergence result more precise it is convenient to take the following facts into account:

- To state and analyze the convergence of the discrete states Y_h it is convenient to extend them to continuous functions $y_h(x,t)$ with respect to the space variable $x = (x_1, x_2)$. This can be done, as in the previous section, in two different ways either by considering a piecewise linear and continuous extension or extending the discrete Fourier expansion of solutions by keeping exactly the same analytic expression. The control V_h has to be extended as well to a function depending on the continuous variable $0 < x_1 < \pi$. This can be done in the same two ways.

- To state the convergence of controls as $h \to 0$ the initial data Y_h^0 in (3.10) have to be chosen in connection with the initial data y^0 of the PDE (3.1). This may be done in several ways. When y^0 is continuous, Y_h^0 can be taken to be the restriction of y^0 to the mesh-points. Otherwise, one can take average values over cells, or simply truncate the Fourier expansion of the continuous initial datum y^0 by taking the first $N \times N$ terms.

This being made precise, the following result holds:

Theorem 3.1. *Let $T > 0$ be any positive control time. Let $y^0 \in L^2(\Omega)$ and Y_h^0 be as above. Then, the null controls V_h for the semi-discrete problem* (3.10) *are uniformly bounded, with respect to h and converge in $L^2(\Gamma_0 \times (0,T))$ towards the null control of the heat equation* (3.1). *The semi-discrete controlled states Y_h also converge to the controlled state y of the heat equation in $L^2(0,T;H^{-1}(\Omega))$ satisfying the null final condition* (3.2).

Remark 3.2. The result is sharp in what concerns the support Γ_0 of the control. Indeed, as pointed out in [30] this result fails when $[\Gamma_0]_h$ is replaced by the set of indices $[\Gamma_0^*]_h$ in which the first node corresponding to the index $j = 1$ is removed. In that case the observability inequality (3.14) fails because of the existence of a non-trivial solution (3.11) such that Φ vanishes on $[\Gamma_0^*]_h$. This is so in fact because of the existence of a non-trivial eigenvector of the discrete Laplacian A_h with eigenvalue $\lambda_h = 4/h^2$, taking alternating values ± 1 along the diagonal and vanishing out of it.

The main elements of the proof of this result are the following. The key point is precisely proving that the observability inequality (3.14) is uniform with respect to the mesh-size $h > 0$. Once this is done standard variational methods allow proving that the controls are uniformly bounded and then passing to the limit as $h \to 0$. We refer to [27] where the same issue was addressed for the heat equation in thin cylindrical domains by similar tools and to [14] where the limit process was described in detail in the context of the finite-difference semi-discrete approximation of the $1-d$ heat equation.

The method of proof of the uniform estimate (3.14) depends heavily on the Fourier decomposition of solutions. To develop it we need some basic facts about the Fourier decomposition of the discrete Laplacian.

The eigenvalue problem associated with the semi-discrete system (3.11) is as follows:

$$\begin{cases} \frac{1}{h^2}\left[4w_{j,k} - w_{j+1,k} - w_{j-1,k} - w_{j,k+1} - w_{j,k-1}\right] = \lambda w_{j,k}, & (j,k) \in \Omega_h, \\ w_{j,k} = 0, & (j,k) \in [\partial\Omega]_h. \end{cases} \tag{3.16}$$

Its spectrum may be computed explicitly:

$$\lambda^{\ell,m}(h) = \frac{4}{h^2}\left[\sin^2\left(\frac{\ell h}{2}\right) + \sin^2\left(\frac{mh}{2}\right)\right] \tag{3.17}$$

$$W^{\ell,m}(h) = w^{\ell,m}(x)\big|_{x=(jh,kh),\, j,k=0,\cdots,N+1} \tag{3.18}$$

for $\ell, m = 1, \dots, N$, where $w^{\ell,m}(x)$ are the eigenfunctions of the continuous Laplacian:

$$w^{\ell,m}(x) = \frac{2}{\pi}\sin(\ell x_1)\sin(m x_2).$$

In particular, in view of (3.18) the eigenvectors of the discrete system (3.16) are simply the restrictions of the eigenfunctions of the continuous Laplacian to the mesh points. Of course, this is a very particular fact that is not true for general domains Ω.

It is also easy to check that

$$\lambda^{\ell,m}(h) \to \lambda^{\ell,m} = \ell^2 + m^2 \quad \text{as } h \to 0 \tag{3.19}$$

for all $\ell, m \geq 1$, where $\lambda^{\ell,m}$ stand for the eigenvalues of the continuous Laplacian. This confirms that the 5-point finite-difference scheme provides a convergent numerical scheme.

The eigenvectors $\{W^{\ell,m}\}_{\ell,m=1,\dots,N}$ constitute an orthonormal basis of $\mathbb{R}^{N\times N}$ with respect to the scalar product

$$\langle f, \widetilde{f}\rangle_h = \left[h^2 \sum_{j,k=1}^{N} f_{j,k}\widetilde{f}_{j,k}\right]^{1/2}, \tag{3.20}$$

associated with the norm (3.15).

The solution of the semi-discrete adjoint system (3.11) can also be easily developed in this basis:

$$\Phi_h(t) = \sum_{\ell,m=1}^{N} a^{\ell,m} e^{-\lambda^{\ell,m}(h)(T-t)} W^{\ell,m} \tag{3.21}$$

where $\{a^{\ell,m}\}$ are the Fourier coefficients of the datum at time $t = T$:

$$\Phi_h^0 = \sum_{\ell,m=1}^{N} a^{\ell,m} W^{\ell,m}, \quad a^{\ell,m} = \langle \Phi_h^0,\ W^{\ell,m} \rangle_h. \tag{3.22}$$

Solutions may also be rewritten in the form

$$\Phi_h(t) = \sum_{\ell=1}^{N} \psi^m(t) \otimes \sigma^m, \tag{3.23}$$

where

$$\sigma^m = \left(\frac{\sqrt{2}}{\sqrt{\pi}} \sin(mkh) \right)_{k=0,\dots,N+1},$$

so that $W^{\ell,m} = \sigma^\ell \otimes \sigma^m$, and each vector-valued function $\psi^m(t) = (\psi_j^m(t))_{j=0,\dots,N+1}$ is a solution of the $1-d$ semi-discrete problem:

$$\begin{cases} \psi_j' - \left[2\psi_j - \psi_{j+1} - \psi_{j-1}\right]/h^2 + \mu^m \psi_j = 0, & j = 1, \dots, N,\ 0 < t < T, \\ \psi_0 = \psi_{N+1} = 0, & 0 < t < T, \\ \psi_j(T) = \psi_j^0, & j = 1, \dots, N, \end{cases} \tag{3.24}$$

where $\mu^m = \frac{4}{h^2} \sin^2\left(\frac{mh}{2}\right)$.

The observability inequality (3.14) is equivalent to proving the $1-d$ analogue for (3.24), uniformly with respect to the index $m \geq 1$, i.e.

$$\|\psi(0)\|_h^2 \leq C \int_0^T \left| \frac{\psi_1}{h} \right|^2 dt, \tag{3.25}$$

for all ψ^0, ψ being the solution of (3.24), with a constant $C > 0$ which is independent of m.

The proof of this $1-d$ uniform estimate can be developed easily following the arguments in [14]. In fact that inequality is an immediate consequence of the explicit form of the spectrum together with a technical result on series of real exponentials that we recall for the sake of completeness. Consider the class $\mathcal{L}(\xi, M)$ constituted by increasing sequences of positive real numbers $\{\nu_j\}_{j\geq 1}$ such that

$$\nu_{j+1} - \nu_j \geq \xi > 0 \quad \text{for all } j \geq 1, \tag{3.26}$$

$$\sum_{k \geq M(\delta)} \frac{1}{\nu_k} \leq \delta \qquad \text{for all } \delta > 0. \tag{3.27}$$

Here ξ is any positive number and $M \colon (0, \infty) \to \mathbb{N}$ is a function such that $M(\delta) \to \infty$ as $\delta \to 0$. Obviously, different values of ξ and M determine different classes of sequences $\mathcal{L}(\xi, M)$. The following holds (see [14]):

Proposition 3.3. *Given a class of sequences $\mathcal{L}(\xi, M)$ and $T > 0$, there exists a positive constant $C > 0$ such that*

$$\int_0^T \Big| \sum_{k=1}^{\infty} a_k e^{-\nu_k t} \Big|^2 dt \geq \frac{C}{\left[\sum_{k\geq 1} 1/\nu_k\right]} \sum_{k\geq 1} \frac{|a_k|^2 e^{-2\nu_k T}}{\nu_k}, \tag{3.28}$$

for all $\{\nu_k\}_{k\geq 1} \in \mathcal{L}(\xi, M)$ and all bounded sequence $\{a_k\}_{k\geq 1}$.

Note that the sequences of eigenvalues of problems (3.24) belong to the same class $\mathcal{L}(\xi, M)$ for all $h > 0$ and $m \geq 1$. Thus, the constant C in (3.28) is uniform and, consequently, (3.24) holds, with an observability constant independent of $h > 0$ and $m \geq 1$ as well.

Remark 3.4. The same result holds for the case in which the control acts as a right hand side external force applied on a band, i.e. on a set of the form $\omega = \{(x_1, x_2) : 0 < x_1 < \gamma, 0 < x_2 < \pi\}$ with $0 < \gamma < \pi$. The corresponding continuous model reads

$$\begin{cases} y_t - \Delta y = f 1_\omega & \text{in } \Omega \times (0, T), \\ y = 0 & \text{on } \partial\Omega \times (0, T), \\ y(x, 0) = y^0(x) & \text{in } \Omega, \end{cases} \tag{3.29}$$

where $f = f(x_1, x_2, t)$ is the control and 1_ω is the characteristic function of the set ω where the control is applied.

The corresponding observability inequality is

$$\|\varphi(0)\|^2_{L^2(\Omega)} \leq C \int_0^T \int_\omega \varphi^2 \, dx dt \quad \text{for all } \varphi^0 \in L^2(\Omega). \tag{3.30}$$

The problems can be formulated similarly for the semi-discrete scheme we have considered.

The observability inequality (3.30) and the corresponding semi-discrete versions hold uniformly with respect to the mesh-size parameter $h > 0$. Consequently the heat equation (3.29) and the corresponding semi-discretizations are uniformly (with respect to $h > 0$) null controllable. Convergence of controls and states holds as well.

In this case the most natural functional setting is the following one. The initial data y^0 belongs to $L^2(\Omega)$, the control f lies in $L^2(\omega \times (0, T))$ and the solutions then belong to $C([0, T]; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$. Convergences hold in these classes as well.

3.3. General domains. The methods of proof of the previous section, based on Fourier series expansions, do not apply to general domains. In fact, even in the context of the continuous heat equation, the existing proofs of null controllability require obtaining the observability estimates by Carleman inequalities (see [5] and [11]). So far the discrete or semi-discrete version of these Carleman inequalities and its possible applications to observability estimates for numerical approximation schemes for the heat equation is a completely open subject of research.

However, in view of the results of the previous section, and using a classical argument, based on extending the control domain and then getting the controls as restrictions to the original boundary of the controlled states, one can derive similar results for general domains but provided *the controls are supported everywhere on the boundary of the domain.* The problem of determining sharp conditions on the subsets of the boundary so that the semi-discrete systems are uniformly controllable is completely open. As we have mentioned above, even in the simplest geometry of the square domain of the previous subsection, the result fails to hold without some restrictions on the support of the control that are not needed for the continuous heat equation.

The following holds:

Theorem 3.5. *For all bounded smooth domain Ω, all time $T > 0$ and all initial data $y^0 \in L^2(\Omega)$, there exists a uniformly bounded sequence of discrete controls $V_h \in L^2(\partial\Omega_h \times (0,T))$ ensuring the null controllability of the finite-difference semi-discrete approximation in Ω_h. These controls can be chosen so that the solutions Y_h converge weakly in $L^2(0,T;H^1(\Omega))$ to the solution y of the heat equation satisfying the null final condition* (3.2).

Proof. Let us briefly explain how this classical extension-restriction method can be implemented in this framework.

Without loss of generality we can assume that Ω is contained in the square domain $\tilde{\Omega} = (0,\pi) \times (0,\pi)$. We discretize the square as in the previous sections, and define the approximating domains Ω_h as those that, having their boundary constituted by mesh-points, better approximate the domain Ω. For the sake of simplicity we assume that Ω_h contains Ω. We also consider a band-like control subdomain ω in the square $\tilde{\Omega}$ so that the results of the previous sections apply and $\Omega_h \cap \omega = \emptyset$ for all $h > 0$.

Given initial data $y^0 \in L^2(\Omega)$ for the continuous heat equation we define approximating discrete data Y_h^0 in Ω_h in a standard way, for instance, by simply taking on each mesh-point the average of y^0 on the neighboring square of sides of size h. This data can be easily extended by zero to discrete data $\tilde{Y}_h^0$ defined in the whole mesh of the square. In view of the results of the previous section (Remark 3.4) this generates controls F_h with support in ω, which are uniformly bounded in $L^2(\omega \times (0,T))$ and converging, as $h \to 0$, to the control of the heat equation (3.29) in the square $\tilde{\Omega}$. This yields also uniformly bounded states $\tilde{Y}_h$ in the space $C([0,T];L^2(\tilde{\Omega})) \cap L^2(0,T;H_0^1(\tilde{\Omega}))$. Obviously, here, as in previous sections, these bounds hold in fact for the piecewise linear continuous extensions of the discrete solutions.

More precisely, the corresponding solutions $\tilde{Y}_h$ converge to the solution y of the heat equation in the space $L^2(0,T;H_0^1(\tilde{\Omega}))$. We can then restrict these solutions to the domains Ω_h and obtain the solutions Y_h of the semi-discrete system in Ω_h, which, by construction, satisfy the final null condition (3.13) and converge to the solution of the heat equation. These solutions satisfy non-homogeneous boundary conditions.

We read their trace as the boundary controls V_h in $\partial\Omega_h$ (resp. $\partial\Omega$) for the semi-discrete (resp. continuous) heat equations. These controls are bounded in $L^2(\partial\Omega_h \times (0, T))$ because they are traces of solutions of bounded energy in $L^2(0, T; H_0^1(\tilde{\Omega}))$. Their weak convergence can also be proved. However, at this point one has to be careful since the controls are defined on boundaries $\partial\Omega_h$ that depend on h. A possible way of stating that convergence rigorously is considering smooth test functions $\theta(x)$ defined everywhere in the square and ensuring that $\int_{\partial\Omega_h} V_h \theta d\sigma$ tends to $\int_{\partial\Omega} v\theta d\sigma$, as $h \to 0$ for all smooth test functions θ. This convergence property of controls holds as well. □

Remark 3.6. The method of proof we have presented based on the extension of the domains and using the previously proved results on the square has two main drawbacks:

- The first one is that the control is required to be supported everywhere on the boundary of the domain. We emphasize however that, despite the fact that no geometric restrictions are needed for the continuous heat equation, in the sense that null controllability holds from an arbitrarily small open subset of the boundary, that is not the case for the semi-discrete one. Thus, the class of subsets of the boundary for which passing to limit on the null-controllability property is possible is still to be clarified, and the result above showing that the whole boundary always suffices is the first positive one in this direction.

- The second one is that it is based on the results obtained in the square by Fourier series techniques. As we have mentioned above, the main tool to deal with continuous heat equations are the Carleman inequalities. As far as we know there is no discrete counterpart of those inequalities and this would be essential to deal with more general heat equations with variable coefficients, or semilinear perturbations. The methods described in Section 2 showing the two-grid controllability of the semilinear wave equation by compactness-uniqueness arguments do not apply for heat-like equations because of their very strong time-irreversibility. Thus, the Carleman approach seems to be the most promising one. However, the fact that observability fails for the semi-discrete system for some observation subdomains indicates that the problem is complex in the sense that the discrete version of the continuous Carleman inequality does not hold. This is a widely open subject of research.

Remark 3.7. Similar results hold for a semi-discrete regular finite-element approximation of the heat equation, as long as solutions can be developed in Fourier series, allowing to reduce the problem in the square to a one-parameter family of $1-d$ problems, and then apply the extension-restriction method to address general domains.

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