# Fractional Brownian motion: stochastic calculus and applications

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**Abstract.** Fractional Brownian motion (fBm) is a centered self-similar Gaussian process with stationary increments, which depends on a parameter $H \in (0, 1)$ called the Hurst index. In this note we will survey some facts about the stochastic calculus with respect to fBm using a pathwise approach and the techniques of the Malliavin calculus. Some applications in turbulence and finance will be discussed.

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## 1. Introduction

A real-valued stochastic process $X = \{X_t, t \geq 0\}$ is a family of random variables

$$X_t : \Omega \to \mathbb{R}$$

defined on a probability space $(\Omega, \mathcal{F}, P)$. The process $X$ is called *Gaussian* if for all $0 \leq t_1 < t_2 < \cdots < t_n$ the probability distribution of the random vector $(X_{t_1}, \ldots, X_{t_n})$ on $\mathbb{R}^n$ is normal or Gaussian. From the properties of the normal distribution it follows that the probability distribution of a Gaussian process is entirely determined by the mean function $\mathbb{E}(X_t)$ and the covariance function

$$\operatorname{Cov}(X_t, X_s) = \mathbb{E}((X_t - \mathbb{E}(X_t))(X_s - \mathbb{E}(X_s))),$$

where $\mathbb{E}$ denotes the mathematical expectation or integral with respect to the probability measure $P$.

One of the most important stochastic processes used in a variety of applications is the *Brownian motion* or *Wiener process* $W = \{W_t, t \geq 0\}$, which is a Gaussian process with zero mean and covariance function $\min(s, t)$. The process $W$ has independent increments and its formal derivative $\frac{dW_t}{dt}$ is used as input noise in dynamical systems, giving rise to stochastic differential equations. The stochastic calculus with respect to the Brownian motion, developed from the works of Itô in the forties, permits to formulate and solve stochastic differential equations.

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Motivated from some applications in hydrology, telecommunications, queueing theory and mathematical finance, there has been a recent interest in input noises without independent increments and possessing long-range dependence and self-similarity properties. Long-range dependence in a stationary time series occurs when the covariances tend to zero like a power function and so slowly that their sums diverge. The self-similarity property means invariance in distribution under a suitable change of scale. One of the simplest stochastic processes which is Gaussian, self-similar and it has stationary increments is fractional Brownian motion, which is a generalization of the classical Brownian motion. As we shall see later, the fractional Brownian motion possesses long-range dependence when its Hurst parameter is larger than 1/2.

In this note we survey some properties of the fractional Brownian motion, and describe different methods to construct a stochastic calculus with respect to this process. We will also discuss some applications in mathematical finance and in turbulence.

## 2. Fractional Brownian motion

A Gaussian process $B^H = \{B_t^H, t \geq 0\}$ is called *fractional Brownian motion* (fBm) of Hurst parameter $H \in (0,1)$ if it has mean zero and the covariance function

$$\mathbb{E}(B_t^H B_s^H) = R_H(t,s) = \frac{1}{2}\big(s^{2H} + t^{2H} - |t-s|^{2H}\big). \tag{2.1}$$

This process was introduced by Kolmogorov [25] and studied by Mandelbrot and Van Ness in [30], where a stochastic integral representation in terms of a standard Brownian motion was established. The parameter $H$ is called Hurst index from the statistical analysis, developed by the climatologist Hurst [24], of the yearly water run-offs of Nile river.

The fractional Brownian motion has the following properties.

1. *Self-similarity:* For any constant $a > 0$, the processes $\{a^{-H} B_{at}^H, t \geq 0\}$ and $\{B_t^H, t \geq 0\}$ have the same probability distribution. This property is an immediate consequence of the fact that the covariance function (2.1) is homogeneous of order $2H$, and it can be considered as a "fractal property" in probability.

2. *Stationary increments:* From (2.1) it follows that the increment of the process in an interval $[s,t]$ has a normal distribution with zero mean and variance

$$\mathbb{E}\big((B_t^H - B_s^H)^2\big) = |t-s|^{2H}. \tag{2.2}$$

Hence, for any integer $k \geq 1$ we have

$$\mathbb{E}\big((B_t^H - B_s^H)^{2k}\big) = \frac{(2k)!}{k!2^k}|t-s|^{2Hk}. \tag{2.3}$$

Choosing $k$ such that $2Hk > 1$, Kolmogorov's continuity criterion and (2.3) imply that there exists a version of the fBm with continuous trajectories. Moreover, using Garsia–Rodemich–Rumsey lemma [19], we can deduce the following modulus of continuity for the trajectories of fBm: For all $\varepsilon > 0$ and $T > 0$, there exists a nonnegative random variable $G_{\varepsilon,T}$ such that $\mathbb{E}(|G_{\varepsilon,T}|^p) < \infty$ for all $p \geq 1$, and, almost surely,

$$\left|B_t^H - B_s^H\right| \leq G_{\varepsilon,T}|t-s|^{H-\varepsilon},$$

for all $s, t \in [0, T]$. In other words, the parameter $H$ controls the regularity of the trajectories, which are Hölder continuous of order $H - \varepsilon$, for any $\varepsilon > 0$.

For $H = 1/2$, the covariance can be written as $R_{1/2}(t,s) = \min(s,t)$, and the process $B^{1/2}$ is an ordinary Brownian motion. In this case the increments of the process in disjoint intervals are independent. However, for $H \neq 1/2$, the increments are not independent.

Set $X_n = B_n^H - B_{n-1}^H$, $n \geq 1$. Then $\{X_n, n \geq 1\}$ is a Gaussian stationary sequence with unit variance and covariance function

$$\begin{aligned}\rho_H(n) &= \frac{1}{2}\left((n+1)^{2H} + (n-1)^{2H} - 2n^{2H}\right)\\ &\approx H(2H-1)n^{2H-2} \to 0,\end{aligned}$$

as $n$ tends to infinity. Therefore, if $H > \frac{1}{2}$, $\rho_H(n) > 0$ for $n$ large enough and $\sum_{n=1}^{\infty} \rho_H(n) = \infty$. We say that the sequence $\{X_n, n \geq 1\}$ has *long-range dependence.* Moreover, this sequence presents an aggregation behavior which can be used to describe cluster phenomena. For $H < \frac{1}{2}$, $\rho_H(n) < 0$ for $n$ large enough and $\sum_{n=1}^{\infty} |\rho_H(n)| < \infty$. In this case, $\{X_n, n \geq 1\}$ can be used to model sequences with intermittency.

**2.1. Construction of the fBm.** In order to show the existence of the fBm we should check that the symmetric function $R_H(t,s)$ defined in (2.1) is nonnegative definite, that is,

$$\sum_{i,j=1}^{n} a_i a_j R_H(t_i, t_j) \geq 0 \tag{2.4}$$

for any sequence of real numbers $a_i$, $i = 1, \ldots, n$ and for any sequence $t_i \geq 0$. Property (2.4) follows from the integral representation

$$B_t^H = \frac{1}{C_1(H)} \int_{\mathbb{R}} \left[((t-s)^+)^{H-\frac{1}{2}} - ((-s)^+)^{H-\frac{1}{2}}\right] dW_s, \tag{2.5}$$

where $\{W(A), A$ Borel subset of $\mathbb{R}\}$ is a Brownian measure on $\mathbb{R}$ and

$$C_1(H) = \left(\int_0^{\infty} \left((1+s)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}}\right)^2 ds + \frac{1}{2H}\right)^{\frac{1}{2}},$$

obtained by Mandelbrot and Van Ness in [30]. The stochastic integral (2.5) is well defined, because the function $f_t(s) = ((t-s)^+)^{H-\frac{1}{2}} - ((-s)^+)^{H-\frac{1}{2}}$, $s \in \mathbb{R}$, $t \geq 0$ satisfies $\int_{\mathbb{R}} f_t(s)^2 ds < \infty$. On the other hand, the right-hand side of (2.5) defines a zero mean Gaussian process such that

$$\mathbb{E}\big((B_t^H)^2\big) = t^{2H}$$

and

$$\mathbb{E}\big((B_t^H - B_s^H)^2\big) = (t-s)^{2H},$$

which implies that $B^H$ is an fBm with Hurst parameter $H$.

**2.2. *p*-variation of the fBm.** Suppose that $X = \{X_t, t \geq 0\}$ is a stochastic process with continuous trajectories. Fix $p > 0$. We define the $p$-variation of $X$ on an interval $[0, T]$ as the following limit in probability:

$$\lim_{n\to\infty} \sum_{j=1}^{n} \big| X_{\frac{jT}{n}} - X_{\frac{(j-1)T}{n}} \big|^p.$$

If the $p$-variation exists and it is nonzero a.s., then for any $q > p$ the $q$-variation is zero and for any $q < p$ the $q$-variation is infinite. For example, the 2-variation (or quadratic variation) of the Brownian motion is equal to the length of the interval $T$.

Rogers has proved in [40] that the fBm $B^H$ has finite $1/H$-variation equals to $c_p T$, where $c_p = \mathbb{E}(|B_1^H|^p)$. In fact, the self-similarity property implies that the sequence

$$\sum_{j=1}^{n} \big| B^H_{\frac{jT}{n}} - B^H_{\frac{(j-1)T}{n}} \big|^{1/H}$$

has the same distribution as

$$\frac{T}{n} \sum_{j=1}^{n} \big| B_j^H - B_{j-1}^H \big|^{1/H},$$

and by the Ergodic Theorem this converges in $L^1(\Omega)$ and almost surely to $\mathbb{E}(|B_1^H|^p)T$.

As a consequence, the fBm with Hurst parameter $H \neq 1/2$ is not a semimartingale. Semimartingales are the natural class of processes for which a stochastic calculus can be developed, and they can be expressed as the sum of a bounded variation process and a local martingale which has finite quadratic variation. The fBm cannot be a semimartingale except in the case $H = 1/2$ because if $H < 1/2$, the quadratic variation is infinite, and if $H > 1/2$ the quadratic variation is zero and the 1-variation is infinite.

Let us mention the following surprising result proved by Cheridito in [8]. Suppose that $\{B_t^H, t \geq 0\}$ is an fBm with Hurst parameter $H \in (0,1)$, and $\{W_t, t \geq 0\}$ is an ordinary Brownian motion. Assume they are independent and set

$$M_t = B_t^H + W_t.$$

Then $\{M_t, t \geq 0\}$ is not a semimartingale if $H \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \frac{3}{4}\right]$, and it is a semimartingale, equivalent in law to a Brownian motion on any finite time interval $[0, T]$, if $H \in \left(\frac{3}{4}, 1\right)$.

The $1/H$-variation of Wick stochastic integrals with respect to the fractional Brownian motion with parameter $H > 1/2$ has been computed by Guerra and Nualart in [20].

## 3. Stochastic calculus with respect to the fBm

The aim of the stochastic calculus is to define stochastic integrals of the form

$$\int_0^T u_t \, dB_t^H, \tag{3.1}$$

where $u = \{u_t, t \in [0, T]\}$ is some stochastic process. If $u$ is a deterministic function there is a general procedure to define the stochastic integral of $u$ with respect to a Gaussian process using the convergence in $L^2(\Omega)$. We will first review this general approach in the particular case of the fBm.

**3.1. Integration of deterministic processes.** Consider an fBm $B^H = \{B_t^H, t \geq 0\}$ with Hurst parameter $H \in (0, 1)$. Fix a time interval $[0, T]$ and denote by $\mathcal{E}$ the set of step functions on $[0, T]$. The integral of a step function of the form

$$\varphi_t = \sum_{j=1}^m a_j \mathbf{1}_{(t_{j-1}, t_j]}(t)$$

is defined in a natural way by

$$\int_0^T \varphi_t \, dB_t^H = \sum_{j=1}^m a_j (B_{t_j}^H - B_{t_{j-1}}^H).$$

We would like to extend this integral to a more general class of functions, using the convergence in $L^2(\Omega)$. To do this we introduce the Hilbert space $\mathcal{H}$ defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathcal{H}} = R_H(t, s).$$

Then the mapping $\varphi \longrightarrow \int_0^T \varphi_t \, dB_t^H$ can be extended to a linear isometry between $\mathcal{H}$ and the Gaussian subspace $H_T(B^H)$ of $L^2(\Omega, \mathcal{F}, P)$ spanned by the random variables $\{B_t^H, t \in [0,T]\}$. We will denote this isometry by $\varphi \longrightarrow B^H(\varphi)$.

We would like to interpret $B^H(\varphi)$ as the stochastic integral of $\varphi \in H_T(B^H)$ with respect to $B^H$ and to write $B^H(\varphi) = \int_0^T \varphi_t dB_t^H$. However, we do not know whether the elements of $\mathcal{H}$ can be considered as real-valued functions. This turns out to be true for $H < \frac{1}{2}$, but is false when $H > \frac{1}{2}$ (see Pipiras and Taqqu [38], [39]). We state without proof the following results about the space $\mathcal{H}$.

**3.1.1. Case $H > \frac{1}{2}$.** In this case the second partial derivative of the covariance function

$$\frac{\partial^2 R_H}{\partial t \partial s} = \alpha_H |t-s|^{2H-2},$$

where $\alpha_H = H(2H-1)$, is integrable, and we can write

$$R_H(t,s) = \alpha_H \int_0^t \int_0^s |r-u|^{2H-2} \, du dr. \tag{3.2}$$

Formula (3.2) implies that the scalar product in the Hilbert space $\mathcal{H}$ can be written as

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = \alpha_H \int_0^T \int_0^T |r-u|^{2H-2} \varphi_r \psi_u \, du dr \tag{3.3}$$

for any pair of step functions $\varphi$ and $\psi$ in $\mathcal{E}$.

As a consequence, we can exhibit a linear space of functions contained in $\mathcal{H}$ in the following way. Let $|\mathcal{H}|$ be the Banach space of measurable functions $\varphi \colon [0,T] \to \mathbb{R}$ such that

$$\|\varphi\|_{|\mathcal{H}|}^2 = \alpha_H \int_0^T \int_0^T |r-u|^{2H-2} |\varphi_r||\varphi_u| \, du dr < \infty.$$

It has been shown in [39] that the space $|\mathcal{H}|$ equipped with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ is not complete and it is isometric to a subspace of $\mathcal{H}$. The following estimate has been proved in [31] using Hölder and Hardy–Littlewood inequalities.

**Lemma 3.1.** *Let $H > \frac{1}{2}$ and $\varphi \in L^{\frac{1}{H}}([0,T])$. Then*

$$\|\varphi\|_{|\mathcal{H}|} \le b_H \|\varphi\|_{L^{\frac{1}{H}}([0,T])}, \tag{3.4}$$

*for some constant $b_H$.*

Thus we have the embeddings

$$L^2([0,T]) \subset L^{\frac{1}{H}}([0,T]) \subset |\mathcal{H}| \subset \mathcal{H},$$

and Wiener-type integral $\int_0^T \varphi_t dB_t$ can be defined for functions $\varphi$ in the Banach space $|\mathcal{H}|$. Notice that we can integrate more functions that in the case of the Brownian motion, and the isometry property of the Itô stochastic integral is replaced here by the formula

$$\mathbb{E}\left(\left(\int_0^T \varphi_t \, dB_t^H\right)^2\right) = \alpha_H \int_0^T \int_0^T |r-u|^{2H-2} \varphi_r \varphi_u \, du dr = \|\varphi\|_{\mathcal{H}}^2.$$

**3.1.2. Case $H < \frac{1}{2}$.** In this case, one can show that $\mathcal{H} = I_{T-}^{\frac{1}{2}-H}(L^2([0,T]))$ (see [14] and Proposition 6 of [2]), where $I_{T-}^{\frac{1}{2}-H}$ is the right-sided fractional integral operator defined by

$$I_{T-}^{H-\frac{1}{2}} \varphi(t) = \frac{1}{\Gamma\left(H-\frac{1}{2}\right)} \int_t^T (s-t)^{H-\frac{3}{2}} \varphi_s \, ds.$$

This means that $\mathcal{H}$ is a space of functions. Moreover the norm of the Hilbert space $\mathcal{H}$ can be computed as follows:

$$\|\varphi\|_{\mathcal{H}}^2 = c_H^2 \int_0^T s^{1-2H} \left(D_{T-}^{\frac{1}{2}-H}\left(u^{H-\frac{1}{2}} \varphi_u\right)\right)^2(s) \, ds, \tag{3.5}$$

where $c_H$ is a constant depending on $H$ and $D_{T-}^{\frac{1}{2}-H}$ is the right-sided fractional derivative operator. The operator $D_{T-}^{\frac{1}{2}-H}$ is the inverse of $I_{T-}^{H-\frac{1}{2}}$, and it has the following integral expression:

$$D_{T-}^{\frac{1}{2}-H} \varphi(t) = \frac{1}{\Gamma\left(H+\frac{1}{2}\right)} \left( \frac{\varphi_t}{(T-t)^{\frac{1}{2}-H}} + \left(\frac{1}{2}-H\right) \int_t^T \frac{\varphi_t - \varphi_s}{(s-t)^{\frac{3}{2}-H}} \, ds \right). \tag{3.6}$$

The following embeddings hold:

$$C^\gamma([0,T]) \subset \mathcal{H} \subset L^{1/H}([0,T])$$

for any $\gamma > H - \frac{1}{2}$. The first inclusion is a direct consequence of formula (3.6), and the second one follows from Hardy–Littlewood inequality. Roughly speaking, in this case the fractional Brownian motion is more irregular than the classical Brownian motion, and some Hölder continuity is required for a function to be integrable. Moreover the computation of the variance of an integral using formula (3.5) is more involved.

**3.2. Integration of random processes.** Different approaches have been used in the literature in order to define stochastic integrals with respect to the fBm. Lin [26] and Dai and Heyde [13] have defined a stochastic integral $\int_0^T u_t dB_t^H$ as limit in $L^2$ of Riemann sums in the case $H > \frac{1}{2}$. The techniques of Malliavin calculus have been used to develop the stochastic calculus for the fBm starting from the pioneering work of Decreusefond and Üstünel [14]. We refer to the works of Carmona and Coutin in [7], Alòs, Mazet and Nualart [1], [2], Alòs and Nualart [3], and the recent monograph by Hu [21], among others. We will first describe a path-wise approach based on Young integrals.

**3.2.1. Path-wise approach.** We can define $\int_0^T u_t \, dB_t^H$ using path-wise Riemann–Stieltjes integrals taking into account the results of Young in [43]. In fact, Young proved that the Riemann–Stieltjes integral $\int_0^T f_t \, dg_t$ exists, provided that $f, g\colon [0,T] \to \mathbb{R}$ are Hölder continuous functions of orders $\alpha$ and $\beta$ with $\alpha + \beta > 1$. Therefore, if $u = \{u_t, t \in [0,T]\}$ is a stochastic process with $\gamma$-Hölder continuous trajectories, where $\gamma > 1 - H$, then the Riemann–Stieltjes integral $\int_0^T u_t \, dB_t^H$ exists path-wise. That is for any element $\omega \in \Omega$, the integral $\int_0^T u_t(\omega) \, dB_t^H(\omega)$ exists as the point-wise limit of Riemann sums. In particular, if $H > 1/2$, the path-wise Riemann–Stieltjes integral $\int_0^T F(B_t^H) \, dB_t^H$ exists if $F$ is a continuously differentiable function. Moreover the following change of variables formula holds:

$$\Phi(B_t^H) = \Phi(0) + \int_0^t F(B_s^H) \, dB_s^H \tag{3.7}$$

if $\Phi' = F$.

In the case $\frac{1}{4} < H < \frac{1}{2}$, there is a path-wise approach to the stochastic integrals of the form

$$\int_0^T F(B_t^H) \, dB_t^H$$

using the theory of *rough paths analysis* introduced by Lyons in [27] (see also [28]). This theory has allowed Coutin and Qian [12] to show the existence of a solution and to prove the convergence of the Wong–Zakai approximations for stochastic differential equations driven by an fBm with Hurst parameter $H \in \left(\frac{1}{4}, \frac{1}{2}\right)$.

Nevertheless, unlike the case of the Itô stochastic integral with respect to the Brownian motion, the path-wise integral $\int_0^T F(B_t^H) \, dB_t^H$ does not have zero mean and there is no easy formula for its variance. We are going to explain how the techniques of Malliavin calculus allow us to compute the mean and the variance of this integral.

**3.3. Malliavin calculus for the fBm.** Let $B^H = \{B_t^H, t \geq 0\}$ be an fBm with Hurst parameter $H \in (0,1)$. The process $B^H$ is Gaussian and we can develop the corresponding stochastic calculus of variations or Malliavin calculus. The Malliavin calculus is an infinite dimensional differential calculus introduced by Malliavin in [29] to provide a probabilistic proof of Hörmander hypoellipticity theorem. The basic operators of Malliavin calculus are the derivative operator $D$ and its adjoint the divergence operator $\delta$. We refer to Nualart [32] and [33] for a detailed account of the Malliavin calculus and its application in the framework of the fBm.

Fix a time interval $[0, T]$. Let $\mathcal{S}$ be the set of elementary random variables of the form

$$F = f(B^H(\varphi_1), \dots, B^H(\varphi_n)), \tag{3.8}$$

where $n \geq 1$, $f \in C_p^\infty(\mathbb{R}^n)$ ($f$ and all its partial derivatives have polynomial growth order), and $\varphi_i \in \mathcal{H}$.

The *derivative operator* $D$ of an elementary random variable $F$ of the form (3.8) is defined as the $\mathcal{H}$-valued random variable

$$DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(B^H(\varphi_1), \dots, B^H(\varphi_n))\varphi_i.$$

The following integration-by-parts formula holds.

**Lemma 3.2.** *Let $F$ be an elementary random variable of the form* (3.8). *Then, for any $\varphi \in \mathcal{H}$ we have*

$$\mathbb{E}(\langle DF, \varphi\rangle_{\mathcal{H}}) = \mathbb{E}(FB^H(\varphi)). \tag{3.9}$$

*Proof.* First notice that we can normalize Eq. (3.9) and assume that the norm of $\varphi$ is one. There exist orthonormal elements of $\mathcal{H}$, $e_1, \dots, e_n$, such that $\varphi = e_1$ and $F$ is an elementary random variable of the form

$$F = f(B^H(e_1), \dots, B^H(e_n)),$$

where $f$ is in $C_p^\infty(\mathbb{R}^n)$. Let $\phi(x)$ denote the density of the standard normal distribution on $\mathbb{R}^n$, that is,

$$\phi(x) = (2\pi)^{-\frac{n}{2}} \exp\Big(-\frac{1}{2}\sum_{i=1}^{n} x_i^2\Big).$$

Then we have

$$\begin{aligned}\mathbb{E}(\langle DF, \varphi\rangle_{\mathcal{H}}) &= \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_1}(x)\phi(x)\,dx \\ &= \int_{\mathbb{R}^n} f(x)\phi(x)x_1\,dx \\ &= \mathbb{E}(FB^H(e_1)) = \mathbb{E}(FB^H(\varphi)),\end{aligned}$$

which completes the proof of the lemma. □

As a consequence, if $F$ and $G$ are elementary random variables and $h \in \mathcal{H}$, then we have

$$\mathbb{E}(G\langle DF, h\rangle_{\mathcal{H}}) = \mathbb{E}(-F\langle DG, h\rangle_{\mathcal{H}} + FGB^H(h)). \tag{3.10}$$

Formula (3.10) implies that the derivative operator $D$ is a closable operator from $L^p(\Omega)$ into $L^p(\Omega; \mathcal{H})$, for any $p \geq 1$. We denote by he Sobolev space $\mathbb{D}^{1,p}$ is the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,p} = [\mathbb{E}(|F|^p) + \mathbb{E}(\|DF\|_{\mathcal{H}}^p)]^{1/p}.$$

One can interpret $\mathbb{D}^{1,p}$ as an infinite-dimensional weighted Sobolev space.

The *divergence operator* $\delta$ is the adjoint of the derivative operator. That is, we say that a random variable $u$ in $L^2(\Omega; \mathcal{H})$ belongs to the domain of the divergence operator, denoted by Dom $\delta$, if

$$|\mathbb{E}(\langle DF, u\rangle_{\mathcal{H}})| \leq c_u \|F\|_{L^2(\Omega)}$$

for any $F \in \mathcal{S}$. In this case $\delta(u)$ is defined by the duality relationship

$$\mathbb{E}(F\delta(u)) = \mathbb{E}(\langle DF, u\rangle_{\mathcal{H}}), \tag{3.11}$$

for any $F \in \mathbb{D}^{1,2}$.

For example, consider an elementary $\mathcal{H}$-valued random variable of the form $u = \sum_{k=1}^m F_k \varphi_k$, where $F_k \in \mathbb{D}^{1,2}$ and $\varphi_k \in \mathcal{H}$. Then, $u$ belongs to the domain of the divergence and from (3.10) we deduce

$$\delta(u) = \sum_{k=1}^m [F_k B^H(\varphi_k) - \langle DF_k, \varphi_k\rangle_{\mathcal{H}}]. \tag{3.12}$$

The expression $F_k B^H(\varphi_k) - \langle DF_k, \varphi_k\rangle_{\mathcal{H}}$ is called the *Wick product* of the random variables $F_k$ and $B^H(\varphi_k)$ and it is denoted by

$$F_k \diamond B^H(\varphi_k) = F_k B^H(\varphi_k) - \langle DF_k, \varphi_k\rangle_{\mathcal{H}}. \tag{3.13}$$

With this notation (3.12) can be written as

$$\delta(u) = \sum_{k=1}^m F_k \diamond B^H(\varphi_k).$$

We will make use of the notation

$$\delta(u) = \int_0^T u_t \diamond dB_t^H,$$

when $u$ is a stochastic process in the domain of the divergence operator.

Here are some basic formulas of the Malliavin calculus which hold for any elementary random variables $F$ and $u$.

$$\mathbb{E}(\delta(u)^2) = \mathbb{E}(\|u\|_{\mathcal{H}}^2) + \mathbb{E}(\langle Du, (Du)^*\rangle_{\mathcal{H}\otimes\mathcal{H}}), \tag{3.14}$$

$$\delta(Fu) = F\delta(u) - \langle DF, u\rangle_{\mathcal{H}}, \tag{3.15}$$

$$\langle D(\delta(u)), h\rangle_{\mathcal{H}} = \langle u, h\rangle_{\mathcal{H}} + \delta(\langle Du, h\rangle_{\mathcal{H}}), \tag{3.16}$$

where $(Du)^*$ is the adjoint of $Du$ in the Hilbert space $\mathcal{H} \otimes \mathcal{H}$. Equation (3.14) holds for any $u$ in the Sobolev space $\mathbb{D}^{1,2}(\mathcal{H})$ of $\mathcal{H}$-valued random variables and it implies that $\mathbb{D}^{1,2}(\mathcal{H}) \subset \operatorname{Dom} \delta$ . Equation (3.15) holds if $F \in \mathbb{D}^{1,2}$, $u$ belongs to the domain of $\delta$ and $Fu$ and $F\delta(u) + \langle DF, u\rangle_{\mathcal{H}}$ are square integrable. Finally, the commutation relation (3.16) holds for any $h \in \mathcal{H}$ and $u \in \mathbb{D}^{1,2}(\mathcal{H})$ such that $\delta(u) \in \mathbb{D}^{1,2}$:

In case of an ordinary Brownian motion, the adapted processes in $L^2([0,T] \times \Omega)$ belong to the domain of the divergence operator, and on this class of processes the divergence operator coincides with the Itô stochastic integral (see Nualart and Pardoux [34]). Actually, the divergence operator coincides with an extension of Itô's stochastic integral introduced by Skorohod in [42]. This is a consequence of formula (3.13), because if $\varphi_k = \mathbf{1}_{[a_k,b_k]}$ and $F_k$ is a random variable measurable with respect to the $\sigma$-field generated by $\{B_t^{1/2}, t \le a_k\}$, then $\langle DF_k, \mathbf{1}_{[a_k,b_k]}\rangle_{L^2([0,T])} = 0$, and the Wick product of $F_k$ and $B_{b_k}^{1/2} - B_{a_k}^{1/2}$ is equal to the ordinary product. Notice here that the random variables $F_k$ and $B_{b_k}^{1/2} - B_{a_k}^{1/2}$ are independent.

**3.4. Wick integrals with respect to the fBm.** A natural question in this framework is to ask in which sense the divergence operator with respect to a fractional Brownian motion $B$ can be interpreted as a stochastic integral. The following proposition provides an answer to this question.

**Proposition 3.3.** *Fix a time interval* $[0,T]$. *Let* $F$ *be a function of class* $C^1$ *such which satisfies, together with* $F'$, *the growth condition*

$$|F(x)| \le c e^{\lambda x^2}, \tag{3.17}$$

*where* $c$ *and* $\lambda$ *are positive constants such that* $\lambda < \frac{1}{4T^{2H}}$. *Suppose that* $H > \frac{1}{2}$. *Then,* $F(B_t^H)$ *belongs to the domain of the divergence operator and*

$$\int_0^T F(B_t^H) \diamond dB_t^H = \int_0^T F(B_t^H)\, dB_t^H - H \int_0^T F'(B_t^H) t^{2H-1}\, dt, \tag{3.18}$$

*where* $\int_0^T F(B_t^H)\, dB_t^H$ *is the path-wise Riemann–Stieltjes integral.*

**Remarks.** 1. Formula (3.18) leads to the following equation for the expectation of a path-wise integral:

$$\mathbb{E}\left( \int_0^T F(B_t^H)\, dB_t^H \right) = H \int_0^T \mathbb{E}(F'(B_t^H)) t^{2H-1}\, dt.$$

2. Suppose that $F$ is a function of class $C^2$ such that $F$, $F'$ and $F''$ satisfy the growth condition (3.17). Then, (3.18) and (3.7) yield

$$F(B_T^H) = F(0) + \int_0^T F'(B_t^H) \diamond dB_t^H + H \int_0^T F''(B_t^H) t^{2H-1}\, dt, \tag{3.19}$$

which can be considered as an Itô formula for the Wick integral.

*Proof of Proposition* 3.3. Set $t_i = \frac{iT}{n}$. Then formula (3.13) yields

$$\sum_{i=1}^{n} F(B^H_{t_{i-1}}) \diamond (B^H_{t_i} - B^H_{t_{i-1}}) = \sum_{i=1}^{n} F(B^H_{t_{i-1}})(B^H_{t_i} - B^H_{t_{i-1}}) - \sum_{i=1}^{n} \langle D(F(B^H_{t_{i-1}})), \mathbf{1}_{[t_{i-1},t_i]}\rangle_{\mathcal{H}}.$$

We have, using the chain rule and $DB^H_{t_{i-1}} = \mathbf{1}_{[0,t_{i-1}]}$,

$$\begin{aligned}\langle D(F(B^H_{t_{i-1}})), \mathbf{1}_{[t_{i-1},t_i]}\rangle_{\mathcal{H}} &= F'(B^H_{t_{i-1}})\langle \mathbf{1}_{[0,t_{i-1}]}, \mathbf{1}_{[t_{i-1},t_i]}\rangle_{\mathcal{H}} \\ &= F'(B^H_{t_{i-1}})(R_H(t_{i-1}, t_i) - R_H(t_{i-1}, t_{i-1})) \\ &= \frac{1}{2} F'(B^H_{t_{i-1}})((t_i)^{2H} - (t_{i-1})^{2H} - (t_i - t_{i-1})^{2H}).\end{aligned}$$

Then it suffices to take the limit as $n$ tends to infinity. The convergences are almost surely and in $L^2(\Omega)$. □

As an application of Proposition 3.3 we will derive the following estimate for the variance of the path-wise stochastic integral of a trigonometric function.

**Proposition 3.4.** *Let $B^H$ be a $d$-dimensional fractional Brownian motion with Hurst parameter $H > 1/2$. Then for any $\xi \in \mathbb{R}^d$ we have*

$$\mathbb{E}\left( \left\| \int_0^T e^{i\langle \xi, B^H_t\rangle} \, dB^H_t \right\|^2_{\mathbb{C}} \right) \le C(1 \wedge |\xi|^{\frac{1}{H}-2}), \tag{3.20}$$

*where $\|z\|_{\mathbb{C}} = \sum_{i=1}^d z^i \bar{z}^i$ and $C$ is a constant depending on $T$, $d$ and $H$.*

*Proof.* From (3.18) we get

$$\int_0^T e^{i\langle \xi, B^H_t\rangle} \, dB^H_t = \int_0^T e^{i\langle \xi, B^H_t\rangle} \diamond dB^H_t + H \int_0^T i\xi e^{i\langle \xi, B^H_t\rangle} t^{2H-1} \, dt. \tag{3.21}$$

We denote by $\pi_\xi(x) = x - \frac{\xi}{|\xi|^2}\langle \xi, x\rangle$ the projection operator on the orthogonal subspace of $\xi$. Clearly

$$\int_0^T e^{i\langle \xi, B^H_t\rangle} \, dB^H_t = \pi_\xi\left( \int_0^T e^{i\langle \xi, B^H_t\rangle} \, dB^H_t \right) + \frac{i\xi}{|\xi|^2}(e^{i\langle \xi, B^H_T\rangle} - 1), \tag{3.22}$$

and, as a consequence, it suffices to show the estimate (3.20) for the first summand in the right-hand side of (3.22). From (3.21) it follows that

$$Z := \pi_\xi\left( \int_0^T e^{i\langle \xi, B^H_t\rangle} \, dB^H_t \right) = \pi_\xi\left( \int_0^T e^{i\langle \xi, B^H_t\rangle} \diamond dB^H_t \right).$$

Then we need to compute the expectation of the square norm of the $\mathbb{C}^3$-valued random variable $Z$. This is done using the duality relationship (3.11) and the commutation formula (3.16). The composition of the projection operator $\pi_\xi$ and the derivative operator $D$ vanishes on a random variable of the form $e^{i\langle \xi, B_t^H\rangle}$. Hence, only the first term in the commutation formula (3.16) applied to $u_t = e^{i\langle \xi, B_t^H\rangle}$ will contribute to $\mathbb{E}(\|Z\|_{\mathbb{C}}^2)$ and we obtain

$$\begin{aligned}\mathbb{E}(\|Z\|_{\mathbb{C}}^2) &= \sum_{j=1}^{d} \mathbb{E}(Z^j \overline{Z^j}) \\ &= \sum_{j=1}^{d} \left(1 - \frac{(\xi^j)^2}{|\xi|^2}\right) \mathbb{E}\left(\langle e^{-i\langle \xi, B_\cdot^H\rangle}, e^{-i\langle \xi, B_\cdot^H\rangle}\rangle_{\mathcal{H}}\right) \\ &= (d-1)\alpha_H \int_0^T \int_0^T \mathbb{E}\left(e^{i\langle \xi, B_s^H - B_r^H\rangle}\right)|s-r|^{2H-2}\, ds dr \\ &= (d-1)\alpha_H \int_0^T \int_0^T e^{-\frac{|s-r|^{2H}}{2}|\xi|^2}|s-r|^{2H-2}\, ds dr,\end{aligned}$$

which leads to the desired estimate. □

Proposition 3.3 also holds for $H \in \left(\frac{1}{4}, \frac{1}{2}\right]$ if we replace the path-wise integral in the right-hand side of (3.18) by the *Stratonovich integral* defined as the limit in probability of symmetric sums

$$\int_0^T F(B_t^H)\, dB_t^H = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2}\left[F\left(B^H_{\frac{(i-1)T}{n}}\right) + F(B^H_{\frac{iT}{n}})\right]\left(B^H_{\frac{iT}{n}} - B^H_{\frac{(i-1)T}{n}}\right).$$

For $H = 1/2$ the Wick integral appearing in Equation (3.18) is the classical Itô integral and it is the limit of forward Wick or ordinary Riemann sums:

$$\begin{aligned}\int_0^T F\left(B_t^{1/2}\right) \diamond dB_t^{1/2} &= \lim_{n\to\infty} \sum_{i=1}^{n} F\left(B^{1/2}_{\frac{(i-1)T}{n}}\right) \diamond \left(B^{1/2}_{\frac{iT}{n}} - B^{1/2}_{\frac{(i-1)T}{n}}\right) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} F\left(B^{1/2}_{\frac{(i-1)T}{n}}\right)\left(B^{1/2}_{\frac{iT}{n}} - B^{1/2}_{\frac{(i-1)T}{n}}\right).\end{aligned}$$

Nevertheless, for $H < 1/2$ the forward Riemann sums do not converge in general. For example, in the simplest case $F(x) = x$, we have, with the notation $t_i = \frac{iT}{n}$

$$\begin{aligned}\mathbb{E}\left(\sum_{i=1}^{n}(B^H_{t_{i-1}}(B^H_{t_i} - B^H_{t_{i-1}}))\right) &= \frac{1}{2}\sum_{i=1}^{n}\left[t_i^{2H} - t_{i-1}^{2H} - (t_i - t_{i-1})^{2H}\right] \\ &= \frac{1}{2}T^{2H}(1 - n^{1-2H}) \to -\infty,\end{aligned}$$

as $n$ tends to infinity.

The convergence of the forward Wick Riemann sums to the forward Wick integral in the case $H \in \left(\frac{1}{4}, \frac{1}{2}\right)$ has been recently established in [36] and [5]. More precisely, the following theorem has been proved in [36].

**Theorem 3.5.** *Suppose $H \in \left(\frac{1}{4}, \frac{1}{2}\right)$ and let $F$ be a function of class $C^7$ such that $F$ together with its derivatives satisfy the growth condition* (3.17). *Then, the forward Wick integral*

$$\int_0^T F(B_t^H) \diamond dB_t^H = \lim_{n\to\infty} \sum_{i=1}^n F\big(B^H_{\frac{(i-1)T}{n}}\big) \diamond \big(B^H_{\frac{iT}{n}} - B^H_{\frac{(i-1)T}{n}}\big)$$

*exists and the Wick–Itô formula* (3.19) *holds.*

More generally, we can replace the fractional Brownian motion $B^H$ by an arbitrary Gaussian process $\{X_t, t \geq 0\}$ with zero mean and continuous covariance function $R(s,t) = \mathbb{E}(X_s X_t)$. Suppose that the variance function $V_t = \mathbb{E}(X_t^2)$ has bounded variation on any finite interval and the following conditions hold for any $T > 0$:

$$\lim_{n\to\infty} \sum_{i,j=1}^n \Big(\mathbb{E}\big(\big(X_{\frac{iT}{n}} - X_{\frac{(i-1)T}{n}}\big)\big(X_{\frac{jT}{n}} - X_{\frac{(j-1)T}{n}}\big)\big)\Big)^2 \to 0, \tag{3.23}$$

$$\lim_{n\to\infty} \sum_{i=1}^n \sup_{0\leq t\leq T} \Big(\mathbb{E}\big(\big(X_{\frac{iT}{n}} - X_{\frac{(i-1)T}{n}}\big) X_t\big)\Big)^2 \to 0. \tag{3.24}$$

Then it is proved in [36] that the forward Wick integral $\int_0^T F(X_t) \diamond dX_t$ exists and the following version of the Wick–Itô formula holds:

$$F(X_T) = F(X_0) + \int_0^T F'(X_t) \diamond dX_t + \frac{1}{2} \int_0^T F''(X_t)\, dV_t.$$

## 4. Application of fBm in turbulence

The observations of three-dimensional turbulent fluids indicate that the vorticity field of the fluid is concentrated along thin structures called vortex filaments. In his book Chorin [10] suggests probabilistic descriptions of vortex filaments by trajectories of self-avoiding walks on a lattice. Flandoli [17] introduced a model of vortex filaments based on a three-dimensional Brownian motion. A basic problem in these models is the computation of the kynetic energy of a given configuration.

Denote by $u(x)$ the velocity field of the fluid at point $x \in \mathbb{R}^3$, and let $\xi = \mathrm{curl} u$ be the associated vorticity field. The kynetic energy of the field will be

$$\mathbb{H} = \frac{1}{2} \int_{\mathbb{R}^3} |u(x)|^2 dx = \frac{1}{8\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\xi(x) \cdot \xi(y)}{|x-y|}\, dx dy. \tag{4.1}$$

We will assume that the vorticity field is concentrated along a thin tube centered in a curve $\gamma = \{\gamma_t, 0 \leq t \leq T\}$. Moreover, we will choose a random model and consider this curve as the trajectory of a three-dimensional fractional Brownian motion $B^H = \{B_t^H, 0 \leq t \leq T\}$ with Hurst parameter $H$. That is, the components of the process $B^H$ are independent fractional Brownian motions. This modelization is justified by the fact that the trajectories of the fractional Brownian motion are Hölder continuous of any order $H \in (0, 1)$. For technical reasons we are going to consider only the case $H > \frac{1}{2}$.

Then the vorticity field can be formally expressed as

$$\xi(x) = \Gamma \int_{\mathbb{R}^3} \left( \int_0^T \delta(x - y - B_s^H) \dot{B}_s^H \, ds \right) \rho\,(dy), \tag{4.2}$$

where $\Gamma$ is a parameter called the circuitation, and $\rho$ is a probability measure on $\mathbb{R}^3$ with compact support.

Substituting (4.2) into (4.1) we derive the following formal expression for the kinetic energy:

$$\mathbb{H} = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathbb{H}_{xy} \rho(dx) \rho(dy), \tag{4.3}$$

where the so-called interaction energy $\mathbb{H}_{xy}$ is given by the double integral

$$\mathbb{H}_{xy} = \frac{\Gamma^2}{8\pi} \sum_{i=1}^{3} \int_0^T \int_0^T \frac{1}{|x + B_t^H - y - B_s^H|} \, dB_s^{H,i} dB_t^{H,i}. \tag{4.4}$$

We are interested in the following problems: Is $\mathbb{H}$ a well defined random variable? Does it have moments of all orders and even exponential moments?

In order to give a rigorous meaning to the double integral (4.4) we introduce the regularization of the function $|\cdot|^{-1}$:

$$\sigma_n = |\cdot|^{-1} * p_{1/n}, \tag{4.5}$$

where $p_{1/n}$ is the Gaussian kernel with variance $\frac{1}{n}$. Then the smoothed interaction energy

$$\mathbb{H}_{xy}^n = \frac{\Gamma^2}{8\pi} \sum_{i=1}^{3} \int_0^T \left( \int_0^T \sigma_n(x + B_t^H - y - B_s^H) \, dB_s^{H,i} \right) dB_t^{H,i} \tag{4.6}$$

is well defined, where the integrals are path-wise Riemann–Stieltjes integrals. Set

$$\mathbb{H}^n = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathbb{H}_{xy}^n \rho(dx) \rho(dy). \tag{4.7}$$

The following result has been proved in [35].

**Theorem 4.1.** *Suppose that the measure $\rho$ satisfies*

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |x-y|^{1-\frac{1}{H}}\rho(dx)\rho(dy) < \infty. \tag{4.8}$$

*Let $\mathbb{H}^n_{xy}$ be the smoothed interaction energy defined by* (4.5). *Then $\mathbb{H}^n$ defined in* (4.7) *converges, for all $k \geq 1$, in $L^k(\Omega)$ to a random variable $\mathbb{H} \geq 0$ that we call the energy associated with the vorticity field* (4.2).

If $H = \frac{1}{2}$, the fBm $B^H$ is a classical three-dimensional Brownian motion. In this case condition (4.8) would be $\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |x-y|^{-1}\rho(dx)\rho(dy) < \infty$, which is the assumption made by Flandoli [17] and Flandoli and Gubinelli [18]. In this last paper, using Fourier approach and Itô's stochastic calculus, the authors show that $\mathbb{E}(e^{-\beta\mathbb{H}}) < \infty$ for sufficiently small negative $\beta$.

The proof of Theorem 4.1 is based on the stochastic calculus with respect to fBm and the application of Fourier transform. Using Fourier transform we can write

$$\frac{1}{|z|} = \int_{\mathbb{R}^3} (2\pi)^3 \frac{e^{-i\langle \xi, z\rangle}}{|\xi|^2}\, d\xi$$

and

$$\sigma_n(x) = \int_{\mathbb{R}^3} |\xi|^{-2} e^{i\langle \xi, x\rangle - |\xi|^2/2n}\, d\xi. \tag{4.9}$$

Substituting (4.9) into in (4.6), we obtain the following formula for the smoothed interaction energy:

$$\begin{aligned}\mathbb{H}^n_{xy} &= \frac{\Gamma^2}{8\pi}\sum_{j=1}^{3}\int_0^T\int_0^T \left(\int_{\mathbb{R}^3} e^{i\langle \xi, x+B_t-y-B_s\rangle}\frac{e^{-|\xi|^2/2n}}{|\xi|^2}\right) dB_s^{H,j}\, dB_t^{H,j} \\ &= \frac{\Gamma^2}{8\pi}\int_{\mathbb{R}^3} |\xi|^{-2} e^{i\langle \xi, x-y\rangle - |\xi|^2/2n} \|Y_\xi\|^2_{\mathbb{C}}\, d\xi,\end{aligned} \tag{4.10}$$

where

$$Y_\xi = \int_0^T e^{i\langle \xi, B_t^H\rangle}\, dB_t^H.$$

Integrating with respect to $\rho$ yields

$$\mathbb{H}^n = \frac{\Gamma^2}{8\pi}\int_{\mathbb{R}^3} \|Y_\xi\|^2_{\mathbb{C}} |\xi|^{-2} |\widehat{\rho}(\xi)|^2 e^{-|\xi|^2/2n} d\xi \geq 0. \tag{4.11}$$

From Fourier analysis and condition (4.8) we know that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |x-y|^{1-\frac{1}{H}}\rho(dx)\rho(dy) = C_H \int_{\mathbb{R}^3} |\widehat{\rho}(\xi)|^2 |\xi|^{\frac{1}{H}-4} d\xi < \infty. \tag{4.12}$$

Then, taking into account (4.12) and (4.11), in order to show the convergence in $L^k(\Omega)$ of $\mathbb{H}^n$ to a random variable $\mathbb{H} \geq 0$ it suffices to check that

$$\mathbb{E}(\|Y_\xi\|_{\mathbb{C}}^{2k}) \leq C_k\big(1 \wedge |\xi|^{k(\frac{1}{H}-2)}\big). \tag{4.13}$$

For $k = 2$ this has been proved in Proposition 3.4. The general case $k \geq 2$ follows by similar arguments making use of the *local nondeterminism property* of fBm (see Berman [4]):

$$\mathrm{Var}\Big(\sum_i (B_{t_i}^H - B_{s_i}^H)\Big) \geq k_H \sum_i (t_i - s_i)^{2H}.$$

## 5. Application to financial mathematics

Fractional Brownian motion has been used to describe the behavior to prices of assets and volatilities in stock markets. The long-range dependence self-similarity properties make this process a suitable model to describe these quantities. We refer to Shiryaev [41] for a general description of the applications of fractional Brownian motion to model financial quantities. We will briefly present in this section two different uses of fBm in mathematical finance.

**5.1. Fractional Black and Scholes model.** It has been proposed by several authors to replace the classical Black and Scholes model which has no memory and is based on the geometric Brownian motion by the so-called *fractional Black and Scholes model*. In this model the market stock price of the risky asset is given by

$$S_t = S_0 \exp\left(\mu t + \sigma B_t^H - \frac{\sigma^2}{2} t^{2H}\right), \tag{5.1}$$

where $B^H$ is an fBm with Hurst parameter $H$, $\mu$ is the mean rate of return and $\sigma > 0$ is the volatility. The price of the non-risky assets at time $t$ is $e^{rt}$, where $r$ is the interest rate.

Consider an investor who starts with some initial endowment $V_0 \geq 0$ and invests in the assets described above. Let $\alpha_t$ be the number of non-risky assets and let $\beta_t$ the number of stocks owned by the investor at time $t$. The couple $(\alpha_t, \beta_t)$, $t \in [0, T]$ is called a *portfolio* and we assume that $\alpha_t$ and $\beta_t$ are stochastic processes. Then the investor's wealth or value of the portfolio at time $t$ is

$$V_t = \alpha_t e^{rt} + \beta_t S_t.$$

We say that the portfolio is *self-financing* if

$$V_t = V_0 + r\int_0^t \alpha_s e^{rs} ds + \int_0^t \beta_s dS_s. \tag{5.2}$$

This means that there is no fresh investment and there is no consumption. We see here that the self-financing condition requires the definition of a stochastic integral with respect to the fBm, and there are two possibilities: path-wise integrals and Wick-type integrals.

The use of path-wise integrals leads to the existence of arbitrage opportunities, which is one of the main drawbacks of the model (5.1). Different authors have proved the existence of arbitrages for the fractional Black and Scholes model (see Rogers [40], Shiryaev [41], and Cheridito [9]). By definition, an arbitrage is a self-financing portfolio which satisfies $V_0 = 0$, $V_T \geq 0$ and $P(V_T > 0) > 0$.

In the case $H > \frac{1}{2}$, one can construct an arbitrage in the following simple way. Suppose, to simplify, that $\mu = r = 0$. Consider the self-financing portfolio defined by

$$\beta_t = S_t - S_0,$$
$$\alpha_t = \int_0^t \beta_s dS_s - \beta_t S_t.$$

This portfolio satisfies $V_0 = 0$ and $V_t = (S_t - S_0)^2 > 0$ for all $t > 0$, and hence it is an arbitrage.

In the classical Black and Scholes model (case $H = \frac{1}{2}$), there exists an equivalent probability measure $Q$ under which $\mu = r$ and the discounted price process $\widetilde{S}_t = e^{-rt} S_t$ is a martingale. Then, the discounted value of a self-financing adapted portfolio satisfying $\mathbb{E}_Q\left(\int_0^T \beta_s^2 \widetilde{S}_s^2 ds\right) < \infty$ is a martingale on the time interval $[0, T]$ given by the Itô stochastic integral

$$\widetilde{V}_t = V_0 + \int_0^t \beta_s d\widetilde{S}_s.$$

As a consequence, $V_t = e^{-r(T-t)}\mathbb{E}_Q(V_T|\mathcal{F}_t)$, and the price of an European option with payoff $G$ at the maturity time $T$ is given by $e^{-r(T-t)}\mathbb{E}_Q(G|\mathcal{F}_t)$. The probability $Q$ is called the martingale measure. In the case $H \neq \frac{1}{2}$, there exist an equivalent probability $Q$ under which $\mu = r$ and $S_t = S_0 \exp(\sigma B_t^H - \frac{\sigma^2}{2} t^{2H})$ has constant expectation. However, $e^{-rt} S_t$ is not a martingale under $Q$.

The existence of arbitrages can be avoided using forward Wick integrals to define the self-financing property (5.2). In fact, using the Wick–Itô formula in (5.1) yields

$$dS_t = \mu S_t dt + \sigma S_t \diamond dB_t^H,$$

and then the self-financing condition (5.2) could be written as

$$V_t = V_0 + \int_0^t (r\alpha_s e^{rs} + \mu\beta_s S_s) ds + \sigma \int_0^t \beta_s S_s \diamond dB_s^H.$$

Applying the stochastic calculus with respect to the Wick integral, Hu and Øksendal in [22], and Elliott and Hoek in [16] have derived the following formula for the value of the call option with payoff $(S_T - K)^+$ at time $t \in [0, T]$:

$$C(t, S_t) = S_t \Phi(y_+) - K e^{-r(T-t)} \Phi(y_-), \tag{5.3}$$

where

$$y_\pm = \left( \ln \frac{S_t}{K} + r(T-t) \pm \frac{\sigma^2 (T^{2H} - t^{2H})}{2} \right) \Big/ \sigma \sqrt{T^{2H} - t^{2H}}. \tag{5.4}$$

In [6] Björk and Hult argue that the definition of a self-financing portfolio using the Wick product is quite restrictive and in [37] Nualart and Taqqu explain the fact that in formula (5.4) only the increment of the variance of the process in the interval $[t, T]$ appears, and extend this formula to price models driven by a general Gaussian process.

**5.2. Stochastic volatility models.** It has been observed that in the classical Black and Scholes model the implied volatility $\sigma_{t,T}^{\mathrm{imp}}$ obtained from formula (5.3) for different options written on the same asset is not constant and heavily depends on the time $t$, the time to maturity $T - t$ and the strike price $S_t$. The $U$-shaped pattern of implied volatilites across different strike prices is called "smile", and it is believed that this and other features as the volatility clustering can be explained by stochastic volatility models. Hull and White have proposed in [23] an option pricing model in which the volatility of the asset price is of the form $\exp(Y_t)$, where $Y_t$ is an Ornstein–Uhlenbeck process.

Consider the following stochastic volatility model based on the fractional Ornstein–Uhlenbeck process. The price of the asset $S_t$ is given by

$$dS_t = \mu S_t dt + \sigma_t S_t dW_t,$$

where $\sigma_t = f(Y_t)$ and $Y_t$ is a fractional Ornstein–Uhlenbeck process:

$$dY_t = \alpha(m - Y_t)dt + \beta_t dB_t^H.$$

The process $W_t$ is an ordinary Brownian motion and $B_t^H$ is a fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$, independent of $W$. Examples of functions $f$ are $f(x) = e^x$ and $f(x) = |x|$.

Comte and Renault studied in [11] this type of stochastic volatility model which introduces long memory and mean reverting in the Hull and White setting. The long-memory property allows this model to capture the well-documented evidence of persistence of the stochastic feature of Black and Scholes implied volatilities, when time to maturity increases.

Hu has proved in [21] the following properties of this model.

1) The market is incomplete and martingale measures are not unique.

2) Set $\gamma_t = (r - \mu)/\sigma_t$ and

$$\frac{dQ}{dP} = \exp\left( \int_0^T \gamma_t \, dW_t - \frac{1}{2} \int_0^T |\gamma_t|^2 \, dt \right).$$

Then $Q$ is the *minimal martingale measure* associated with $P$.

3) The risk minimizing-hedging price at time $t = 0$ of an European call option with payoff $(S_T - K)^+$ is given by

$$C_0 = e^{-rT}\mathbb{E}_Q[(S_T - K)^+]. \tag{5.5}$$

As a consequence of (5.5), if $\mathcal{G}_t$ denotes the filtration generated by fBm, we obtain

$$\begin{aligned} C_0 &= e^{-rT}\mathbb{E}_Q[\mathbb{E}_Q((S_T - K)^+|\mathcal{G}_T)] \\ &= e^{-rT}\mathbb{E}_Q[C_{BS}(\sigma)]. \end{aligned}$$

Here $\sigma = \sqrt{\int_0^T \sigma_s^2 ds}$ and $C_{BS}(\sigma)$ is the Black and Scholes price function given by

$$C_{BS} = S_0\Phi(y_+) - Ke^{-rT}\Phi(y_-),$$

where

$$y_\pm = \frac{\ln\frac{S_0}{K} + (r \pm \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}.$$

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