Panel A
Controversial issues in K-12 mathematical education

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Abstract. This article sets the background for the panel session at the ICM on controversial issues in K-12 mathematics education. Three specific issues have been selected: Technology, skill building and the role of test and assessment. For each of these, a list of questions has been prepared. After introducing the three themes and the associated questions, this article presents the positions on these of the two panelists: Professor Anthony Ralston, from the State University of New York at Buffalo in the US, and Professor Ehud de Shalit from the Hebrew University of Jerusalem in Israel. The article ends with some personal comments from the coordinator of the panel: Professor Michèle Artigue from the University Paris 7 in France.

Mathematics Subject Classification (2000). 97A80, 97D30, 97D40, 97D60.

Keywords. Mathematics education, K-12 curriculum, technology, skill building, concept building, testing, assessment.

Introduction

by *Michèle Artigue*

K-12 mathematics education is obviously a controversial area, so much so that, in countries like the US, the term Math Wars has been used for describing the kind of conflicts between communities that has been generated in recent years. We all regularly hear colleagues complaining that the students they receive have not been adequately trained and that, every year, the situation becomes worse, or that they are not pleased with the kind of mathematics education their children receive etc. We all know that such feelings are not something new, but we cannot deny that in the last decade they have dramatically increased in intensity in many countries.

Why does such a situation exist? What are the real challenges that K-12 mathematics education has to face at the beginning of the XXIst century? What can mathematicians do in order to enhance or support efficiently the necessary efforts, evolutions and changes of the whole educational community? These are the crucial issues that motivate the existence of a panel session on Controversial Issues in K-12 mathematical education at the ICM2006 in Madrid. It is certainly interesting to keep them in mind even if the panel does not address them all directly.

Proceedings of the International Congress
of Mathematicians, Madrid, Spain, 2006

For structuring this panel session, we have selected some particularly controversial issues, and will try to elaborate on these, with the support of the audience. These issues approach the current problems met by K-12 mathematics education through three different, but not independent, topics: technology, the place given to the learning of skills and techniques, and assessment and tests. Everyone will certainly agree that each of these is today a controversial topic, and that frequently in what we read or hear, it is advocated that the ways they have been dealt with in recent years or currently has resulted in some of the difficulties in K-12 mathematics education today.

In what follows, we briefly introduce these three topics and articulate some questions that we would like to discuss for each of them. We then present the positions on these questions of the two panelists, Professor Anthony Ralston from the State University of New York at Buffalo, and Professor Ehud de Shalit from the Hebrew University in Jerusalem. The article ends with some general comments by the moderator of this panel session, Professor Michèle Artigue from the University Paris 7.

Topics and questions

Technology. In 1985, the first study launched by ICMI entitled "The influence of computers and informatics on mathematics and its teaching" was devoted to computers and the ways the learning and the teaching of mathematics as well as this discipline itself was affected by technology. A second edition of the book issued from this study was prepared by B. Cornu and A. Ralston and published in 1992 in the Science and Education Series of UNESCO. As described in its introduction, the UNESCO book addresses the importance of the changes introduced by technology in professional mathematical practices and makes suggestions for new curriculum elements based on these new methods of doing mathematics. It is pointed out that even if these suggestions are judged by the reader to be stimulating and even persuasive as well as reasonably grounded, it is nevertheless the case that "such suggestions are fundamentally speculative at the level of large scale implementation – by which we mean that converting them into a well-developed and tested curriculum for the typical teacher and the typical student is still a major challenge."

Since that time, more and more sophisticated technological tools have continued to be developed for supporting the learning and teaching of mathematics, and their use is today encouraged by the K-12 mathematics curriculum in most countries. Nevertheless, in spite of the existence of an increasing amount of positive small-scale experiments, the real nature of the effect of technology on mathematics education in the large remains under discussion. The problems raised in the first ICMI study have not been solved, and the discourse of those who think that the impact of technology is globally negative and ask for a strict limitation of the use of calculators and software, and even for their banishment from mathematics education in the early grades, is opposed by those who consider that it does not make sense today to think about mathematics learning and teaching without taking into account the existence of technology and without trying to benefit from the real and increasing potential it offers for mathematics education.

Thus the first set of questions we propose to raise is:

Up to what point should the changes introduced in social and professional mathematical practices by technology be reflected in mathematics education?

What does technology have to offer today to K-12 mathematics education and why does it seem so difficult to have it benefit mathematics education in the large outside experimental settings?

What could be done in order to improve the current situation?

Is a strict limitation on the use of calculators and software a reasonable solution?

Skill building. Every one of us certainly agrees that mathematical learning, as with any kind of human apprenticeship, requires skill building and also that it requires much more than that. In recent decades all over the world, K-12 mathematics curriculum developers, influenced by constructivist and socio-constructivist epistemologies of learning, by the results of cognitive research on learning processes, and also by the observed limitations of students' achievements in mathematics, have stressed the necessity of moving some distance from teaching practices seen as too focused on drill and practice, and of getting a better balance between the technical and conceptual facets of mathematical learning. K-12 mathematics curricula have given increasing importance to exploration and work on rich and open problems in order to help students understand better the reasons for mathematical conceptualizations, and these conceptualizations themselves. They have also promoted teaching strategies that try to give more importance to the personal and collective elaborations of students in the development of classroom mathematical knowledge. Once more, the global effects of these curricular changes on K-12 mathematics education are a matter of controversy. Voices have arisen asking for a radical change in the role to be given to the learning and mastery of algorithms, with the long division algorithm often appearing as emblematic of the desired changes. In a similar vein, other voices denounce the dangers of what they see as a new "back to basics" program and the inability to understand that mathematics teaching has to take into account social and technological evolution, and the changes in scientific and mathematics culture needed in our societies today.

Thus a second set of questions:

What is the pertinence of the opposition between skill learning and the exploration of rich problems? Between techniques and concepts?

What is the right balance to be achieved in K-12 mathematics education between the different facets of mathematical activity?

How can this balance be achieved and what are the respective mathematics responsibilities to be given to the teachers and the students?

Test and assessment. We are all aware of the influence that the form and the content of assessment have on any form of education and, thus, on K-12 mathematics education. We are also aware of the increasing importance given to national and international testing, as reflected for instance by the coverage in the media of the PISA enterprise of the OECD and TIMSS, and the influence that these results are taking in educational policies. The importance to be given to external assessment versus internal assessment, to international comparisons and standardized testing, to the effect of assessment on the mathematics learning of students, and to the effect of systematic testing on educational systems are all controversial issues, as are the discussions generated by the "No Child Left Behind" legislation in the US. Thus our third set of questions:

How can we correctly reflect in assessment what we wish to achieve through mathematics education?

Is standardized testing ever useful? For what purpose? Under what conditions?

*What exactly is tested by international assessments such as PISA or TIMSS? Do they represent the mathematical culture that we want K-*12 *mathematics education to develop? What can we learn from them?*

A reform perspective

by *Anthony Ralston*

Preamble. I believe passionately that the K-12 mathematical curriculum, as it exists in most countries, needs substantial reform. But, because the notion of "reform curriculum" means different things to different people, I think I should begin by delineating the perspective from which I view the reform of mathematics curricula.

First, neither constructivism nor its antithesis plays any role in my beliefs about reform. Thus, arguments about such things as discovery learning or about whether rote memorization is a good or a bad thing will play no role in what follows here.

Next, I believe strongly that mathematics should be a demanding subject in all grades, probably the most demanding that students study in each grade. Thus, any suggestion that mathematics should be "dumbed down" at any level is anathema to me.

Finally, I believe, as surely all attendees of ICM2006 do, that mathematics is a dynamic, growing subject with ever-changing opinions on what is more important or less important mathematical subject matter. But, perhaps in contradistinction to many ICM attendees, I think this perspective must include not just areas of research but also the entire K-12 curriculum. Thus, what is important subject matter in K-12 mathematics today may be – I think, is – different from what it was yesterday and no doubt is different from what it will be tomorrow.

Technology. Mathematicians were slower than almost all scientists and engineers to make computing technology a part of their everyday working lives[1]. Nowadays, how-

[1]Mathematicians' attitudes about technology as well as about other matters considered in this paper are discussed in A. Ralston, Research Mathematicians and Mathematics Education: A Critique, *Notices Amer. Math. Soc.* **51** (2004), 403–411.

ever, many research mathematicians use computers routinely for number crunching, for accessing computer algebra systems, and for using a variety of other computer software for both professional and non-professional purposes. Still, it appears that, even as most mathematicians now recognize computer technology as an indispensable tool for doing mathematics research, they resist the notion that computers should be widely used in mathematics education on the grounds that what is important in K-12 mathematics education has hardly changed in – dare one say it? – the past century.

The crucial aspect of whether – and, if so, when – computers or calculators should be used in K-12 mathematics education has resulted in more controversy than any other aspect of mathematics education. I have written elsewhere about my belief that pencil-and-paper arithmetic (p-and-p, hereafter) should be abolished from the primary school curriculum in the sense that no level of proficiency in it should be expected of students although teachers should be free to use p-and-p examples as they wish. Since I published a paper to this effect in 1999[2], I have seen no reason, cogent to me, to back off from this position[3]. Of course, you must understand that, keeping in mind the position stated in the Preamble, I would replace a p-and-p-based curriculum with a rigorous curriculum emphasizing mental arithmetic while allowing free use of calculators in all grades. The goal of such a curriculum, as with any arithmetic curriculum in primary school, would be to achieve the *number sense* in students that would enable them to proceed successfully with secondary school mathematics.

I cannot provide any evidence why a mental arithmetic, calculator based curriculum would work because it has not been tried but neither has anyone adduced a compelling reason why it should not work. Moreover, no one can give good reasons to continue the classical p-and-p curriculum which has never worked very well and must now be working more poorly than ever, given that almost all students will recognize that the classical curriculum tries to teach them a skill without practical value any longer. In addition, since students will almost universally use calculators outside the classroom, forbidding them inside the classroom is self-defeating. Only if it can be argued that a p-and-p-based curriculum is clearly the best way to prepare students for subsequent study of mathematics, can such a curriculum be justified in the 21st century. But I don't believe any compelling argument of this nature can be made; all such attempts I've seen can only be described as feeble.

Learning *arithmetic* – what the operations are, when to use them, place value etc. – is crucial for the study of all subsequent mathematics. But not only is p-and-p calculation not necessary to the goal of learning about arithmetic, it is positively destructive of that goal.

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[2]A. Ralston, Let's Abolish Pencil-and-Paper Arithmetic, *Journal of Computers in Mathematics and Science Teaching* **18** (1999), 173–194.

[3]An area of particular controversy is whether the traditional long division algorithm should be taught at all. My opinion on this can be found in A. Ralston, The Case Against Long Division, http://www.doc.ic.ac.uk/∼ar9/LDApaper2.html.

Skill building. Skill building is of value in K-12 math education only insofar as the skills learned facilitate the doing of mathematics and the subsequent study of mathematics. It must be recognized that (almost?) none of the skills traditionally taught in K-12 mathematics have value any longer as skills per se. But, following the foregoing argument, if p-and-p skills are not to be taught, it is imperative that learning substantial mental arithmetic skills should be a major goal of primary school mathematics. These skills should include not just the obvious ones of immediate recall of the addition and multiplication tables and the ability to do all one-digit arithmetic mentally but also the ability to do substantial amounts of two-digit arithmetic mentally.

It needs to be emphasized that the development of good mental arithmetic skills requires good coaching from a teacher about the various algorithms that can be used to do mental arithmetic and then hard work by the student. Mental arithmetic, say two-digit by two-digit multiplication, is hard[4]. Learning to do it well involves much practice during which the student will decide which algorithm is most congenial to her/him. Teaching and learning mental arithmetic must be a joint responsibility of teacher and student.

One advantage of learning to do two-digit arithmetic mentally is that such a skill requires a good grasp of place value, an important aspect of primary school mathematics in any case. Another advantage is that automaticity or near automaticity in one- and two-digit mental arithmetic allows students to be given demanding word problems. More generally, sound technique in mathematics must always be the forerunner of good conceptual understanding.

A word about fractions. Primary school is certainly the place where students should learn about fractions, reciprocals and the conversion of fractions to decimals and vice versa. But I doubt it is the right place for them to learn fraction arithmetic except perhaps in some simple cases. When students get to secondary school, they will need to do arithmetic on algebraic fractions. This would be the best time to teach the arithmetic of both numeric and algebraic fractions since, in any case, few students will remember the arithmetic of numeric fractions from when it may have been taught in primary school.

Test and assessment. The standardized testing culture that has swept over the United States and is rapidly advancing in the United Kingdom and other countries is perhaps the most serious threat of all to quality mathematics education throughout the world. The standardized testing requirements in the U. S. No Child Left Behind (NCLB) legislation will have the almost certain result that NCLB will be that act most destructive of quality education ever passed by the United States Congress.

The pressure on schools and teachers for students to achieve high grades on stan-

[4]Is there any reason why learning to perform two-digit by two-digit multiplication mentally should not be a realizable goal of school mathematics? I don't think so. Some positive evidence is contained in D. Zhang, Some Characteristics of Mathematics Education in East Asia – An Overview from China, in *Proceedings of the Seventh Southeast Asian Conference on Mathematics Education* (N. D. Tri et al., eds.), Vietnamese Mathematical Society, Hanoi, 1997.

dardized tests always leads to a number of evils that have been widely catalogued. Three of the worst are teaching to the test, emphasis on routine mathematics at the expense of advanced topics and problem solving, and the inordinate amount of time taken to prepare for these tests which not only drives important mathematics from the classroom but also often means decreased attention to science, history and the arts generally. Moreover, the inevitable result of emphasis on standardized tests is that scores increase without any concomitant increase in learning[5].

I am not opposed to testing students. Quite the contrary. It is by far the best way for a teacher to assess the learning of her/his students. But in the not quite antediluvian past, the assessment task was left to individual teachers in their classrooms. Why have things changed so much? The answer in the United States and other countries appears to be that educational administrators, politicians and even parents no longer trust classroom teachers to do the assessment job themselves. This is not altogether wrongheaded. As I and others have argued elsewhere[6], the quality of K-12 mathematics teachers in, at least, American schools has been declining for half a century and, while there are still many excellent mathematics teachers in American schools, too many are not competent to teach the mathematics they are supposed to teach[7]. But, if this is so, standardized testing will only exacerbate this problem by convincing too many who might become teachers that there is no scope for imagination or initiative in school mathematics teaching.

The crucial point is that there is no sign whatever that standardized testing has ever been effective in increasing student learning. If all standardized testing in all subjects were abandoned at all levels short of university entrance, this would be an immediate boon to all education.

I should say a word about TIMSS and PISA. Since both of these are essentially diagnostic tools given to a sampling of students, they do not suffer from most of the strictures above. For example, teachers cannot teach to the test because at most a very few students in each class will take these tests.

A traditional perspective

by *Ehud de Shalit*

The author of this essay is a mathematician who found himself involved in questions of mathematical education despite lack of formal background in the discipline. I make

[5]See A.Ralston, The Next Disaster in American Education. *The Sacramento Bee*, 1 December 2002 (http://www/doc.ic.ac.uk/∼ar9/NextDisaster.html).

[6]See A. Ralston, The Real Scandal in American School Mathematics, *Education Week*, 27 April 2005 (http://www/doc.ic.ac.uk/∼ar9/TeacherQual.html) and V.Troen and K.C.Boles, *Who's Teaching Your Children? Why The Teacher Crisis is Worse Than You Think and What Can Be Done about It*, 2003, Yale University Press.

[7]Indeed, while mathematicians generally choose to argue about something we may be knowledgeable about – curriculum – a far more serious problem with mathematics education in most countries is the inability to attract enough high quality people to teach school mathematics.

no claim to know the literature of science education, and I am surely ignorant of important studies in the area. I nevertheless dare to participate in the discussion because I believe that educators and scientists alike should bear the burden of shaping our children's education, listening to and learning from each other's point of view. It is deplorable that recently, the two communities of math educators and mathematicians have been poised against each other, mostly, but not always, the first being portrayed as "reformers", the latter as "traditionalists"[8]. Emotions have run high, and the two communities found themselves in conflict, instead of joining forces towards a common cause.

This being said, I also want to apologize for not having equally strong opinions on all issues. In fact, I will address two of the points raised by Prof. Artigue (*the impact of technology* and *skill building*), and make only minor remarks on the third (*tests and assessment*), which I consider to be a political issue more than a mathematical or educational one. I hope to make myself clear in due time. Moreover, depending on the circumstances, these three sample topics, important as they be, need not have a decisive affect on the success or failure of a given system. External factors such as class size, discipline, teacher training and resources, which vary considerably from state to state, are often of greater importance than questions of curriculum and methodology. However, unable to influence the first in a direct way, we, mathematicians, focus on the latter.

My starting point is that mathematics teaching *need not* necessarily follow the rapid changes in the usage of the subject in society or technology. Its prime role is to imbed in our children a basic sense for, and understanding of numbers, symbols[9], shapes and other "mathematical objects", together with skills in manipulating these objects, that are needed to develop what is commonly called "mathematical reasoning". The objects to be chosen, the time devoted, and what is taught about them, should be dictated by their prominence in mathematics, and their epistemic and pedagogical value, and less so by their frequency in daily life. This does not mean, of course, that examples and applications of the material should not be updated and modernized, but I do preach respect for the traditional way of teaching, because more than it was based on old *needs*, it was based on inherent *values* that have not changed with time. A well-trained mathematical mind is a highly flexible system. If brought up correctly, it will find its way to adjust and analyze mathematical scenarios very different from the ones that surrounded it initially, while it was being shaped.

As an example, consider the well trodden issue of long-division. I believe that the standard algorithm should be taught in elementary school, thoroughly explained and practiced *not* because of its practical value. Rather, it is important because it enhances the understanding of the decimal system, of zero as a place-holder, of the Euclidean algorithm, and is a necessary precursor for polynomial arithmetic. It allows

[8]Those unaware of the ongoing controversies, can read David Ross' article *Math Wars* (www.ios.org/articles/dross_math-wars.asp) and the references therein.

[9]A. Arcavi, Symbol sense: informal sense-making in formal mathematics. *For the Learning of Mathematics* **14** (3) (1994), 24–35.

the child to review the multiplication table and develop number sense while doing something else, more advanced, so it makes learning more interesting. Moreover, it is natural. It therefore agrees with *mental arithmetic*, and helps us visualize the process involved in division. For these and for many other reasons, well explained in[10] and not mentioned here for lack of time, long division is a pedagogical gold mine. The abandoned algorithm for extracting the square root, often cited to justify abolishing long division as well, is in comparison a pedagogical swamp, was abandoned for this reason, and not because it became obsolete.

Respect for traditional values in education has another advantage, that new theories are tested gradually, and radical potentially damaging changes are avoided. An ailing educational system need not be ailing because its underlying principles or methods are old-fashioned, and *reform* in itself is not an automatic cure, even where needed. More than often, the reason for failure is that good old principles stopped being implemented correctly, for various sociological reasons on which I do not want to elaborate here.

The second general remark is that I do not believe in teaching in vacuum, or in a content-empty environment. Learning must focus on concrete concepts, methods, algorithms if necessary. Insight and creativity come with variations on a theme, not where there is no theme. Teaching "how to solve it" is not synonymous with dry cookbook mathematics. It can be fun and enlightening. Constructivism[11] has led some educators to minimize teacher's intervention in the learning process. Such an approach may be tried on a single-time basis, through enrichment activities. But it is time consuming, with the average teacher may lead to fixation of mistakes, and for anyone but the brightest students can be very frustrating. We simply cannot expect the children to come up with the great discoveries of arithmetic and geometry, let alone calculus, by pure exploration. A fundamental feature that distinguishes human beings from animals is that we can learn not only from our own experience, but also from that of our ancestors. To be illustrative, I think of the art of teaching as give-and-take. The teacher delivers a package of knowledge, bit by bit, each time taking back from the students their responses, their reflections, their mistakes. On these she or he builds up, shaping and manipulating the dialogue, until a deep understanding and the desired proficiency are achieved. To believe that these can spring up spontaneously, just by setting the stage and giving a slight stimulus, is to assume too much.

Finally, a word about the term *conceptual thinking*. It is often brought up by advocators of certain approaches in education to distinguish their goals from those of others, who – so it is to be understood – lead to lower level thinking. I don't know of any kind of thinking that is not conceptual. Abstraction, in language or in mathematics, making generalizations, or conversely, looking for examples, testing predictions and searching for the right vocabulary to communicate our mental processes, are all instances of conceptual thinking, namely thinking in terms of concepts. The contro-

[10] *The role of long division in the K-12 curriculum,* by D. Klein and J. Milgram, ftp://math.stanford.edu/pub/papers/milgram/long-division/longdivsiondone.htm.

[11] *Constructivism* is the cognitive theory based on the idea that knowledge is constructed by the learner.

versy, in my view, is not about whether conceptual thinking is more or less important than basic skills, but whether acquiring those skills is part of conceptual thinking, as I want to argue, or not[12].

Skill building. Drill and practice. Like a swimmer or a pianist the student of mathematics has to absorb great ideas, but also to practice hard to be able to use them efficiently. Contrary to the common belief, the primary reason for skill building is not the need to perform mathematical tasks with great precision and speed, because in our age these human qualities have been surpassed by machines, and we need not regret it.

I see three important reasons to promote skill building. The first is that skill building is essential for forming a sense for numbers, and later on for symbols, functions, or geometry. Subtle instances of insight and analogy, are woven into a web of images and associations in one's mind, and cannot be classified and taught sequentially. They are only the product of long-term practicing and skill building. The distance between knowing something in principle and mastering it is very big in mathematics.

The second reason is that our mind functions on several levels simultaneously, and we are not always aware of the sub-conscious levels that are "running in the background", if I may use a metaphor from computer science. To be able to free the thinking creative part of our mind, to let it form the web of links needed for exploration and discovery, we must defer to the background more routine tasks, that in the past occupied the front, but should now be performed semi-automatically.

The last reason in favor of skill building is rarely mentioned, and might seem to you heretical. Experience has taught me that many children, especially those suffering from math phobia or learning disabilities, are highly rewarded psychologically by success in performing a routine algorithm, such as long division, and by acquiring proficiency in a given task. Such a reward for them is a higher boost than the ability to understand the theory behind it, or the fun in discovering a method by themselves. Once they know the "how" they are lead to ask "why". I would not rule out an approach that harnesses skill building before understanding, if the teacher feels that it suits the child better. Needless to say, both aspects should eventually be covered, and bright children who have mastered the technique and eagerly ask good questions should not be hindered.

Skill building is often confined – by those promoting "conceptual understanding" as a substitute – to algorithmic skills, and algorithmic skills are then downgraded to mere rote. While algorithmic skills are very important, and the algorithmic approach to arithmetic is something to be cherished, as I made clear in the example of long division, mathematical skills are by no means only algorithmic. The ability to translate a word problem into arithmetic, or later on into algebra, is a well-defined skill. Analyzed closely, it consists of many sub-skills, like distinguishing relevant information from irrelevant data, choosing the variables cleverly, translating prose into algebra, and

[12] See *Basic skills versus conceptual understanding, a bogus dichotomy in mathematics education,* by H. Wu (http://www.aft.org/pubs-reports/american_educator/fall99/wu.pdf).

finally the technique of solving, say, a system of linear equations. Geometric skills, drawing to scale, recognizing hidden parts, decomposing and assembling figures, as well as computational skills of area and volume, form another category.

Given my earlier criticism of the constructivist approach, it will not come as a surprise that I believe in *standard algorithms*. It is true, students who come up with their own (correct!) algorithms should never be scolded, but eventually standard algorithms are more efficient, help in the process of automatization of algorithmic tasks discussed above, and also serve an important purpose of establishing a common language.

As an example, after a certain amount of preparatory classes meant to clarify the distributive law, which may include both manipulations of brackets and geometric representation by rectangles, I would simply *teach* the standard algorithm for "vertical multiplication". I do not see the benefit in letting the students make up their own algorithms, where inevitably many will multiply units with units, tens with tens etc. and then add them up. To expect from fourth graders to come up with what was one of the main achievements of the Hindus and the Arab scholars in the Middle Ages is unrealistic. However, once the algorithm has been explained, both the *how* and the *why*, and practiced, there are many subtle questions that can be left for discussion and discovery. Would it always be more economical to apply the algorithm as is, or perhaps switching the position of the two numbers to be multiplied saves some operations? How can we estimate in advance the order of magnitude to save us from potential pitfalls, what double-checks should we make etc. etc.

Anthony Ralston, in his paper "*Let's abolish pencil and paper arithmetic*"[13] advocates to abolish basic algorithmic skills that were the bread-and-butter of elementary school arithmetic for centuries. He summarizes his discussion by saying "*Since no one argues any longer that knowledge of PPA* (*pencil-and-paper arithmetic*) *is a useful skill in life* (*or, for that matter, in mathematics*)*, the question is only whether such 'deprivation' could leave students without the understanding or technique necessary to study further mathematics.*"

Even if we accept the premises, doubtful in my mind, I think he misses the point. First, any attempt to separate understanding from technique is artificial. Second, it is the miracle of the subject that the very same principles underlying higher mathematics, or fashionable topics such as geometry and statistics, often quoted as benefitting from the time freed by the abolishment of PPA, are manifested in their purest and simplest form in these basic skills. A person not knowing how to calculate what $\frac{3}{5}$ of $4\frac{2}{7}$kg of rice are, will not have the technique to analyze the changes in the school budget of England. Nor will he have developed enough intimacy with numbers to estimate those changes in advance, or tell instantly, if his calculator-based computations make sense or not.

As a substitute to PPA, Ralston elevates *mental arithmetic* to a central position in his proposed program. To give examples, he expects elementary school students to

[13] In *Journal of Computers in Mathematics and Science Teaching* **18** (2) (1999), 173–194.

perform two-digit by two-digit multiplication mentally, and high-school students "to be able to factor a variety of three term quadratics mentally". To succeed, he admits, mental arithmetic should be practiced in calculator-free environment. I wholeheartedly agree with the importance of mental mathematics, both for developing number (and symbol) sense, and for practical purposes, estimation and checks. I do not understand though the reluctance to allow one to put things on paper. PPA does not contradict mental arithmetic. It records it, something we shouldn't be ashamed of, and without which we cannot communicate or analyze peacefully what we have done. It also helps in visualizing graphically the steps carried in our mind, and it allows us to organize little mental steps into a larger procedure, without putting too heavy a burden on our memory.

The impact of technology. There are two somewhat separate questions here. The first is to what extent should the curriculum be dictated by the way mathematics is used in technology, and to what extent should we conform to requests coming from the changing society, rather than teach basic principles and skills[14]. I have expressed my opinion about this question in the opening statement. Contrary to the quotation just mentioned, I believe that education in the large, ought to enrich the child and teach him or her basic skills, knowledge, values and understanding that are *absolute*. If carried out correctly, they will inevitably produce a knowledgeable, thinking, skilled and creative citizen. If tailored to the needs of a certain industry or society, rather to these absolute values, they will produce poor technocrats.

The second question involved in the issue of technology is to what extent do technological innovations influence the way we teach in class. This concerns mostly calculators in elementary school, but also the use of graphic calculators in calculus, Excel sheets in statistics and computers in general.

It would be wrong to ignore the changes in technology, the challenges that they bring about, and the opportunities which they provide for demonstration and practice. However, we should clearly define our *mathematical goals*, phrase them in mathematical terms and avoid as much as possible slogans, even if we agree with their general mood. We should distinguish mathematical goals from *educational goals*. Only then may we look at issues of technology, and decide whether they help steering math education the right way, or not. To understand the effect this *process* of analyzing the role of technology has, consider the following example.

The child will be able to derive qualitative and quantitative information form graphs such as a graph displaying the change of temperature with altitude.

I hope everybody agrees with the statement as a basic goal of K-12 mathematical education. The terms *qualitative and quantitative information* demand further elaboration, but I shall not go into it. Now suppose we have to choose between graphic

[14]Judah Schwartz, in his essay *Intellectually stimulating and socially responsible school curricula – can technology help us get there?* writes: "By far the dominant expectation of education in most societies, at least as articulated by political leaders and by the print and electronic press, is to prepare people for the world of work."

calculators and pencil-and-paper, for a first encounter with graphs as a tool to communicate observations and measurements. Have we phrased our goal as *The child will learn to appreciate the use of graphs in natural sciences such as climatology,* we might be inclined to favor graphic calculators. They are attractive, have the fragrance of modernism, and provide vast opportunities that pencil and paper do not provide. But are they as good in conveying first principles? Can the child learn from them where to choose to draw the axes, what scale to use, and how to plot the data? Even the mere physical act, the hand-eye coordination in handling the ruler, is fundamental in my eyes to the learning process. Feeding the data into a calculator, then pressing a button, produces wonderful results, but has its pedagogical drawbacks. This does not mean I would discard graphic calculators. At a later stage they can be helpful in adding visual affects that are difficult to achieve without them – zooming in and out, changing scale, flipping the axes, to name a few. I would simply be careful in my choices, which tool to apply first in class.

While I can see the benefits of graphic calculators in middle-school in studying functional dependence, I am much less excited by the use of ordinary calculators in elementary school arithmetic. At this early stage building number sense is the teacher's number one task. I still have to hear one good argument in favor of calculators in this regard. I need no proof for how destructive they can be. Even those opposed to PPA value mental arithmetic, as means for estimation and double-check. Unfortunately we have witnessed all around us, at school and at the university, a significant decline in these skills over the last two decades, that I can only attribute to the introduction of calculators. Whoever agrees that skill building is an important component of mathematical understanding, and cannot be separated from conceptual thinking, must also confess that calculators at an early age are impeding normal mathematical development.

Those advocating early use of calculators necessarily advocate early emphasis on decimals at the expense of simple fraction arithmetic. Is it right? From the point of view of technology, simple fractions are probably obsolete. From the point of view of their pedagogical value, in understanding basic principles of arithmetic, such as ratio and proportionality, or unique factorization, and in anticipating similar structures in algebra, they are indispensable. For all these reasons I would happily ban the use of calculators in class until a solid understanding of arithmetic has been achieved, and the associated skills have been built. I am not in a position to judge whether these happen at the end of fifth, sixth or seventh grade, but the general spirit is clear to me.

Two arguments that are often heard in favor of technology at school are (a) that to oppose it is a lost battle and (b) that technological skills are so important in society, that not teaching them early would deprive certain children, especially those coming from poor families, of future opportunities. To the first argument I have nothing to say, except that if we adopt it we shouldn't be here today. As for the second, I must admit I am very sensitive to the social obligations of educators. Fortunately or unfortunately, home computers are not anymore the sign of a privileged family, much as TV is not a sign of progress, and I honestly believe that mastering Excel carries no more mathematical value than mastering a microwave manual.

Finally, a comment on a growing trend among educators to write computer-assisted material or use sophisticated software, such as Dynamic Geometry Software, in conjunction with the standard curriculum. Some of it is very well made, enriches the learning environment, and I have no objection to computers per se. But from the little I have seen in this medium, in terms of cost-benefit analysis, the added value is not big, so I will never substitute a computer for the informal contact with a talented teacher. When it comes to political decisions, where to invest the money, my preferences are clear, at least in the country I come from.

To this one should add that computers are *not just a tool* to convey the same message more efficiently. Learning in a computerized environment affects our perception of the objects of study. Good or bad, this has to be analyzed before a new computer-dependent program is adopted.

Tests and assessment. Testing is a controversial issue among educators. There is a whole separate session at ICM2006 devoted to two competing international comparative tests – PISA and TIMSS. It is well known that certain educators detest testing altogether, while others build their whole curriculum around it. The more I think on it the more I become convinced that testing is a *political issue,* namely an issue that has to be decided by policy makers, based on an ideology, and taking into account factors that are only remotely related to math education. An excellent example is the controversy around US government act "No Child Left Behind" from 2002.

Testing takes various shapes. It can be comprehensive or diagnostic. You may test accumulated knowledge, or you may test the potential of a student. You may test algorithmic skills, or you may test insight and creativity. (Even though, as I said above, the former are indispensable for developing the latter, when it comes to testing, they are quite different.) A math test can be phrased in formal language or in prose. A test can be confined to one school, to one state, or to a nation. Studies show that the framework within which a problem is set affects the rate of success, and this effect changes with gender and origin. I have not mentioned more radical views which claim that western societies test only "western intelligence", and blame the relative failure of certain minorities on the dominant western frame of mind.

Testing can also serve a variety of goals. It may be purely informative, or can serve to rank, for purpose of admissions or stipends. It can test the students, but it can also test teachers success, and inform them of potential problems. Testing can be used for comparing alternative programs, or it may be needed to impose discipline on students, and on educators.

I regard all these goals as legitimate, and every kind of test welcome, *provided* one knows what kind of information to expect form it. A company recruiting civil engineers will probably test different mathematical skills than a software developer, and a matriculation exam summarizing the achievements of a student in high school need not be similar to entrance exams at a university, where a greater emphasis may be put on the student's potential and creativity.

Obviously teaching should center on the subject-matter and not only prepare for tests, but a change in curriculum often requires frequent testing to make sure the message gets across. Where there is a good tradition, and little intervention is needed, testing can be kept at a minimum. Under different circumstances tests may become a central integral part of the program.

Mathematical education will benefit from an open discussion of the issues raised here and others. It is important that mathematicians will express their views, paying respect to educators, and share their convictions with them. It is important to get to the bottom of examples, and refrain from vague statements. It is important to let changes happen, with ample time if needed, but refrain from changes that are made for the sake of reform alone. Changes must be gradual, and objectively followed. Most new ideas succeed when pushed vigorously with a small group of dedicated teachers, and with a fat budget. The problem is what happens when a case-study involving a dozen schools is over, and those ideas are adopted across the board. Do they carry enough weight to keep the momentum? Are the teachers qualified to spread the gospel?

Concluding remarks

by *Michèle Artigue*

In this panel session, we focus on only a few of many possible controversial issues: technology, skill building, test and assessment. For each theme, as the coordinator of this panel, I articulated a short list of questions and asked the two panelists to express their positions. As could be expected, these positions are quite different, as they probably would be on the following fundamental issues: What do we want to achieve today through elementary and secondary mathematics education? What mathematics should be taught in order to achieve these goals? And how should we teach this mathematics? What are the relationships between mathematics education and the society at large?

K-12 mathematics education does not serve a unique goal. It aims at the transmission from one generation to the next one of a cultural heritage, which is one of the great achievements of humankind, and at the development of the logical reasoning competence which is so strongly attached to it. It aims at providing students with efficient means for understanding the world in which they live, and play their proper role in it. It aims at preparing and making possible the training of future mathematicians and scientists who will be in charge of the development of mathematics and scientific knowledge, and of the teachers who will have the responsibility of the transmission of this knowledge. Such ambitions can be seen as general invariants, but what is certainly not an invariant is the way we understand each of these components, at a given moment, in a given context; the way we understand the adequate balance between these, and last but not least what we consider the most appropriate strategies for achieving these ambitions. Educational systems try to adapt to this variation mainly through curricular changes. The turbulence and controversies we regularly observe attest to the difficulty of this adaptation, and also the fact that the curricular lever chosen is not necessarily the best one.

As a mathematician who has worked in the area of mathematics education for more than 20 years now, I am struck by the simplistic way in which the complex problems that K-12 mathematics education faces today are often approached; the existing tendency to give the same value to rough affirmations and anecdotes as to well founded analysis and discussions; the persistent belief in the existence of easy and immediate solutions; the brutality of the changes imposed on educational systems, without considering their real cost, and without developing the necessary means for understanding observed success and failure. Education in the large seems a world where opposition and slogans are in front of the stage, hiding shades of meaning and dialectic visions. Slogans used by those favoring or opposing the use of technology, opposing positions on the development of concepts and of techniques are typical examples of these. Even educational research, in its attempts to reach a larger audience, does not always avoid undue simplifications and oppositions[15].

For improving the current situation, we need to overcome such a state, and will try to do so in the ICM panel associated with this contribution. But in order to solve the complex and difficult problems that K-12 mathematics education faces today in many countries, we need to do more than express well-articulated positions on controversial issues and the rationale for these. We need coherent and long-term programs, taking into account the specificities of the different contexts and the existing material and human resources. We need exchanges on our respective situations and experiences for improving these, being aware that solutions in mathematics education are always local ones in terms both of space and time, that it is nearly impossible to determine what is the exact field of validity of a given observed result, the field of extension of a given regularity. We need the collaboration of all those who are involved in mathematics education: mathematicians, mathematics educators, teachers and teachers educators, each of whom can contribute different kinds of expertise. One of the ambitions of ICMI, through its series of ICMI Studies, is to foster such exchanges among all those interested in mathematics education and to make clear what is the state of the international reflection on some selected critical issues, what has been achieved and what is needed[16].

I would like to add to these short comments that curricular choices are certainly important but that the dynamics of complex systems, such as educational systems, is not just a matter of curricular choices. The quality of teachers and of teacher education, both pre-service and also in-service, is certainly as important if not more

[15] See for instance M. Artigue, Learning Mathematics in a CAS environment: The Genesis of a Reflection About Instrumentation and the Dialectics Between Technical and Conceptual Work, *International Journal of Computers for Mathematics Learning* **7** (2002), 245–274.

[16] Themes for the most recent ICMI studies have been: The teaching and learning of mathematics at university level, the future of the teaching and learning of algebra, mathematics education in different cultural traditions – a comparative study of East Asia and the West, applications and modelling in mathematics education, the professional education and development of future teachers of mathematics.

important than curricular choices. From this point of view, the fact that mathematics educational research, which has for a long time focused on students, has in the last decade paid increasing attention to the teacher and to teacher education, is a promising evolution. Research tries today to understand the coherence underlying observed teachers' practices[17] the kind of precise mathematical knowledge the profession requires, how it can be developed, how this mathematical knowledge interacts with other forms of professional knowledge, and how these complex interactions influence teachers' practices and students' learning. Interesting results begin to be obtained, which at the same time help us understand better what can be realistic dynamics for change. The final success of the enterprise requires the collaboration of those with diverse expertise[18].

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[17] A. Robert and J. Rogalski, Le système complexe et cohérent des pratiques des enseignants de mathématiques: une double approche, *La revue canadienne des sciences, des mathématiques et des technologies* **2.4** (2002), 505–528.

[18] An example of such a collaboration is given by the Mathematics and Sciences Research Institute in Berkeley which has created an education advisory board and organizes workshops involving mathematicians, mathematics educators, teachers, policy makers etc. on critical issues. The themes of the first two were the assessment of students' mathematical knowledge and the mathematics knowledge for K-8 teachers.