Panel C

The role of mathematicians in K-12 mathematics education

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Shiu-Yuen Cheng, Konrad Osterwalder, and Hung-Hsi Wu (*panelists*)

Abstract. The need for mathematics educators, schoolteachers and mathematicians to work together to improve K-12 mathematics education continues to be a great concern throughout the world. The main paper for this panel discussion proposes a paradigm or perspective within which to organize this working together of the different groups. This is to see mathematics education as mathematical engineering. From this perspective, the challenge of the mathematics educator and the schoolteacher is to customize mathematics to students' needs. The role, in turn, of the university mathematician is to customize mathematics courses for teachers so that they in turn may be able to customize the mathematics for the different needs of their students. This paradigm is then discussed in different contexts, in the United States, Hong Kong, Switzerland, and the Philippines. The paradigm is seen to be fruitful in these different contexts.

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Introduction and overview

by *Ben Nebres, S. J.*

The theme of this panel discussion is "How mathematicians contribute to K-12 mathematics education." Three distinguished mathematicians, coming from different contexts and different mathematics education traditions, provide challenging and helpful insights into this theme. Because the contexts in which they write are quite different (United States, China, Switzerland), it was decided to present the papers separately. This introductory note is meant to highlight the main proposal and perspective coming from Prof. Wu's paper and relate the contributions from the other two panelists to it. I also add a few comments from the context of a developing country, the Philippines.

First, a brief note on the diversity of contexts. In terms of mathematics achievement based on international comparative studies such as TIMSS, the United States ranks in the middle, while Hong Kong and Switzerland rank towards the top. The Philippines ranks towards the bottom. In terms of educational systems, the U.S. is quite decentralized with great diversity in terms of curriculum, textbooks, teacher

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training, while Hong Kong schools would have greater commonality in terms of curriculum, textbooks, assessment. Some would say that it may be better to compare not the performance of all U.S. schools, but to take account of the diversity of systems and compare blocks of schools (by states or groups of school districts). In terms of resources, the U.S., Switzerland and Hong Kong have first world resources, while Philippine schools operate in the context of great scarcity: classes of 80 students in rooms built for 40, one textbook shared by 5 or 6 pupils and so forth. One can even look at the differences in mathematics education between Hong Kong and the U.S. and Switzerland and the Philippines from the point of view of mathematics education cultures. This is discussed in the recently published "Mathematics Education in Different Cultural Traditions: A Comparative Study of East-Asia and the West", edited by Frederick Leung, Klaus Dieter Graf and Frances Lopez-Real, Volume 9 in the New ICMI Study Series. My own role as a mathematician in helping improve mathematics education in the Philippine context of poverty of resources is described in a chapter entitled "Philippine Perspective on the ICMI Comparative Study" in this volume of the ICMI Study Series.

Despite this diversity of contexts, there is agreement in all the three papers (and in the Philippine experience as well) on the importance of the role of mathematicians in K-12 mathematics education and on a particular paradigm or perspective (mathematics education as mathematical engineering) on how mathematicians can effectively fulfill this role.

In the main paper for this panel presentation and discussion, Prof. Hung-Hsi Wu of the University of California Berkeley, proposes a re-conceptualization of mathematics education as mathematical engineering: "Thus chemical engineering is the science of customizing chemistry to solve human problems… . I will put forth the contention that mathematics education is mathematical engineering, in the sense that it is the application of basic mathematical principles to meet the needs of teachers and students." This is somewhat different from the suggestion of Hyman Bass to look at mathematics education as a branch of applied mathematics. In engineering, what is important is the customization of scientific principles to address human needs. Similarly, in mathematics education as mathematical engineering, what is crucial is the customization of mathematical principles to address the needs of teachers and pupils.

From this viewpoint, the challenge is to work out the role of mathematicians in mathematics education analogous to that of physicists in engineering. Just as the roles of physicists and engineers in engineering are deeply intertwined, so should the roles of mathematicians and mathematics educators be in mathematics education. Right now the two worlds are separate and do not communicate well with each other. Prof. Wu writes: "… if mathematicians want to participate in serious educational work in K-12, … the most important thing is the awareness that K-12 mathematics education is not a subset of mathematics, and that there is quite a bit to learn about the process of customization that distinguishes K-12 mathematics education from mathematics."

In my communication with Prof. Wu, we agreed that it is important that the term "mathematics educator" include both the university mathematics educator as researcher and the school mathematics teacher as practitioner. While the university mathematics education researcher is an expert on teaching and learning theories, the mathematics master teacher is most knowledgeable about actual teacher, student, and classroom contexts. Success in improving mathematics education will require good communication and working together among mathematicians, university mathematics educators and school mathematics teachers.

To properly customize mathematics in different student contexts, a mathematics teacher needs: solid mathematical knowledge, clear perception of the setting defined by students' knowledge, and flexibility of mind to customize this mathematics knowledge for use in this particular setting. In this model of mathematics education as mathematical engineering, the role of the mathematician is to provide the solid mathematical knowledge. This should be done in such a way that the teacher is provided with different ways of understanding and approaching a mathematics concept so that he can have a repertoire to draw from in customizing the mathematics for different student contexts. Prof. Wu gives examples such as in the teaching of fractions or in providing intervention for students at-risk.

The paper of Prof. Shiu-Yuen Cheng of the Hong Kong University of Science and Technology picks up from the "mathematics education as mathematical engineering" framework of Prof. Wu and sets it in the context of mathematics education in Hong Kong. He notes "the main factors for providing an effective mathematics education as curriculum design, teacher competence and assessment methods." He says that most important is teacher competence and it is to this factor that mathematicians can contribute the most. They can contribute in the university curriculum for mathematics teacher programs and in in-service workshops for mathematics teachers. Together with Prof. Wu (and from my experience as well) he points out that the university curriculum for mathematics teachers, which is usually a combination of courses for mathematics majors and education courses, "do not serve the purpose of providing the necessary understanding to be a competent mathematics teacher." (Reasons for this are very well argued in Liping Ma, "Knowing and Teaching Elementary Mathematics.")

Prof. Cheng points out that the Hong Kong mathematics education context is one which shows great success as shown by the excellent performance of Hong Kong students in international comparative studies. "In Hong Kong, Johnny can add! In fact, Johnny can do fractions and decimals quite well." There is, however, a downside to this achievement. Prof. Cheng is concerned that this is at great cost, particularly in "suffocating students' creativity and motivation for learning." He stresses the important role of mathematicians in communicating effectively to the public and to decision-makers these important concerns for mathematics education. (This balance between effective mastery of fundamentals and the need to foster creativity has been an important recent concern in East Asia and was the theme of the ICMI-East Asian Regional Conference on Mathematics Education in Shanghai in August 2005.)

Prof. K. Osterwalder of the ETH Zurich writes in the context of the upper years of the Swiss gymnasium (years 9–12) and the role of mathematicians in universities such as ETH Zurich in preparing mathematics teachers for these upper years. He points out that in Switzerland, students do quite well in international comparative studies in mathematics. Teachers are well trained. They are required to get a masters degree at a level where they could equally opt to go into industry as mathematicians. On the role of mathematicians in K-12 mathematics education, he agrees that the main contribution of research mathematicians is in the education of mathematics teachers. He focuses in a special way on the "Specialized Mathematics Courses with an Educational Focus" taken by mathematics teachers in the university. He provides various examples of course material, from linear equations and linear algebra to noting recent research breakthroughs accessible to gymnasium students, where these course materials "narrow the gap between Gymnasium mathematics and University mathematics" in the spirit of Felix Klein's "Elementary Mathematics from a Higher Viewpoint."

The paradigm or perspective of "mathematics education as mathematical engineering" proposed by Prof. Wu is thus seen to be quite fruitful in these different contexts. They all point to the central role of the mathematics teacher, whose challenge is to customize the mathematics to the students' needs. The important role of the university mathematician is then to customize mathematics courses for these teachers in such a way that they may in turn be able to customize the mathematics needed by students in different contexts and with different needs.

How mathematicians can contribute to K-12 mathematics education

by *Hung-Hsi Wu*

> *"To overcome the isolation of education research, more effective links must be created between educational faculties and the faculties of universities. This could allow scholars of education better acquaintance with new developments in and across the disciplines and other professional fields of the university, while also encouraging discipline-based scholars with interests in education to collaborate in the study of education."*
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> Lagemann [14], p. 241.

I would like to make a general disclaimer at the outset. I think I should only talk about things I know firsthand, so I will limit my comments to the K-12 mathematics education in the U.S. rather than take a more global view. Such a restriction is not necessarily fatal since a friend of mine observed that what takes place in the U.S. tends also to take place elsewhere a few years later. For example, in France there is now a Math War that resembles the American Math Wars of the nineties (Education Week [7]). We live in a global village after all.

Let me begin with a fairy tale. Two villages are separated by a hill, and it was decided that for ease of contact, they would drill a tunnel. Each village was entrusted with the drilling of its own half of the tunnel, but after both had done their work, it was discovered that the two halves didn't meet in the middle of the hill. Even though a connecting tunnel between the two lengths already built could be done at relatively small expense, the two villages, each in defense of its honor, prefer to continue the quarrel to this day.

This fairy tale is too close to reality for comfort when the two villages are replaced by the education and mathematics communities, with the former emphasizing the overriding importance of pedagogy and the latter, mathematical content.[1] Mathematics education rests on the twin pillars of mathematics and pedagogy, but the ongoing saga in mathematics education is mostly a series of episodes pitting one against the other. There is probably no better proof of the disunity between these communities than the very title of this article. Indeed, if someone were to write about "How chemists can contribute to chemical engineering", that person would be considered a crank for wasting ink on a non-issue. Chemical engineering is a well-defined discipline, and chemical engineers are perfectly capable of doing what they are entrusted to do. They know the chemistry they need for their work, and if there is any doubt, they would freely consult with their colleagues in chemistry in the spirit of cooperation and collegiality. Therefore, the fact that we are going to discuss "How mathematicians can contribute to K-12 mathematics education" in the setting of the International Congress speaks volumes about both mathematics education and mathematicians.

In matters of education it is of course natural for the power structure to hold the reins, just as in matters of engineering they are held by engineers. But while the chemical engineers are glad to have chemists down the hall, and glad to learn what they can use in their work, the corresponding relationship has not been the case for mathematics educators. Since education research is thriving and research funding is ample, it is not surprising that educators want to protect their intellectual independence in the university environment. Rumblings about how mathematically unqualified teachers or deficient curricula are undercutting mathematics learning do surface from time to time, but we have not witnessed the expected aggressive action agitating for collaboration with mathematicians. Other troubling issues related to mathematics content, such as the presence of incorrect assessment items in standardized tests, likewise fail to arouse genuine concern in the mathematics education community. To an outsider, the protection of the "education" enclave seems to matter more to university educators than collaboration with the research mathematics community that could strengthen K-12 mathematics education. By contrast, if the department of chemical engineering consistently produces engineers with a defective knowledge of chemistry, or if accidents occur in its laboratories with regular frequency, would the chemical engineering faculty not immediately spring to action? This question prompts the thought that maybe we no longer know what mathematics education is

[1] In writing about sociological phenomena, especially education, it is understood that all statements are statistical in nature unless stated to the contrary, and that exceptions are part and parcel to each statement. In fact, there are striking (though isolated) exceptions in the present context. The reader is asked to be aware of this caveat for the rest of this article.

about and it is time for us to take a second look.

One meaning of the word “engineering” is the art or science of customizing scientific theory to meet human needs. Thus chemical engineering is the science of customizing chemistry to solve human problems, or electrical engineering is the science of customizing electromagnetic theory to design all the nice gadgets that we have come to consider indispensable. I will put forth the contention that mathematics education is mathematical engineering, in the sense that it is the customization of basic mathematical principles to meet the needs of teachers and students.[2] I will try to convince you that this is a good model for the understanding of mathematics education before proceeding to a discussion of how mathematicians can contribute to K-12 mathematics education. The far-from-surprising conclusion is that, unless mathematicians and educators can work as equal partners, K-12 mathematics education cannot improve.

Regarding the nature of mathematics education, Bass made a similar suggestion in [5] that it should be considered a branch of applied mathematics.[3] What I would like to emphasize is the aspect of engineering that customizes scientific principles to the needs of humanity in contrast with the scientific-application aspect of applied mathematics. Thus, when H. Hertz demonstrated the possibility of broadcasting and receiving electromagnetic waves, he made a breakthrough in science by making a scientific application of Maxwell’s theory. But when G. Marconi makes use of Hertz’s discovery to create a radio, Marconi was making a fundamental contribution in electrical engineering, because he had taken the extra step of harnessing an abstract phenomenon to fill a human need.[4] In this sense what separates mathematics education as mathematical engineering from mathematics education as applied mathematics is the crucial step of customizing the mathematics, rather than simply applying it in a straightforward manner to the specific needs of the classroom. There is no better illustration of this idea of customization than the teaching of fractions in upper elementary and middle schools, as I now explain.

Students’ failure to learn fractions is well-known. School texts usually present a fraction as parts of a whole, i.e., pieces of a pizza, and this is the most basic conception of a fraction for most elementary students. However, when fractions are applied to

[2]After the completion of this article, Skip Fennell brought to my attention the article “Access and Opportunities to Learn Are Not Accidents: Engineering Mathematical Progress in Your School” by William F. Tate, which is available at: http://www.serve.org/_downloads/publications/AccessAndOpportunities.pdf. Tate is concerned with equity and uses “engineering” as a metaphor to emphasize the potential for designing different educational policies and pedagogical activities to promote learning, but without addressing the mathematics. On the other hand, the present article explains why mathematics education is the engineering of mathematics.

[3]Hy Bass lectured on this idea in December of 1996 at MSRI, but [5] seems to be a convenient reference. After the completion of this article, Zalman Usiskin informed me that in the Proceedings of the U.S.-Japan workshop on the mathematics education of teachers in 2000 that followed ICME-9 in Japan, he had written that “ ‘Teachers’ mathematics’ is a field of applied mathematics that deserves its own place in the curriculum.” Along this line, let it be mentioned that the paper of Ferrini-Mundy and Findell [8] made the same assertion and, like Bass, it does not touch on the engineering aspect of mathematics education. The need for mathematicians and educators to work on equal footing in mathematics education is likewise not mentioned by these educators.

[4]The invention was actually due to N. Tesla, but like many things in life, popular preception displaces the truth. I am indebted to S. Simic for pointing this out to me.

everyday situations, then it is clear that there is more to fractions than parts-of-a-whole, e.g., if there are 15 boys and 18 girls in a classroom, then the ratio of boys to girls is the fraction, which has nothing to do with cutting up a pizza into 18 equal parts and taking 15. In the primary grades, it is not a serious problem if students' knowledge of fractions is imprecise and informal, so that a fraction can be simultaneously parts-of-a-whole, a ratio, a division, and an operator[5], and a number. Children at that age are probably not given to doubts about the improbability of an object having so many wondrous attributes. At some stage of their mathematical development, however, they will have to make sense of these different "personalities" of a fraction. It is this transition from intuitive knowledge to a more formal and abstract kind of mathematical knowledge that causes the most learning problems. This transition usually takes place in grades 5–7.

There is by now copious mathematics education research[6] on how to facilitate children's learning of the fraction concept at this critical juncture in order to optimize their ability to use fractions efficiently. At present, what most children get from their classroom instruction on fractions is a fragmented picture of a fraction with all these different "personalities" lurking around and coming forward seemingly randomly. What a large part of this research does is to address this fragmentation by emphasizing the cognitive connections between these "personalities". It does so by helping children construct their intuitive knowledge of the different "personalities" of a fraction through the use of problems, hands-on activities, and contextual presentations.

This is a good first step, and yet, if we think through students' mathematical needs beyond grade 7, then we may come to the conclusion that establishing cognitive connections does not go far enough. What students need is an unambiguous definition of a fraction which tells them what a fraction really is. They also need to be exposed to direct, mathematical, connections between this definition and the other "personalities" of a fraction. They have to learn that mathematics is simple and understandable, in the sense that if they can hold onto one clear meaning of a fraction and can reason for themselves, then they can learn all about fractions without ever being surprised by any of these other "personalities".

From a mathematician's perspective, this scenario of having to develop a concept with multiple interpretations is all too familiar. In college courses, one approaches rational numbers (both positive and negative fractions) either abstractly as the prime field of characteristic zero, or as the field of quotients of the integers. The problem is that neither is suitable for use with fifth graders. This fact is recognized by mathematics education researchers, as is the fact that from such a precise and abstract definition of rational numbers, one can prove all the assorted "personalities" of rational numbers. If I have read the research literature correctly, these researchers despair of ever being able to offer proofs once they are forced to operate without an abstract definition, and

[5]For example, the fraction can be regarded as a function (operator) which associates to each quantity three-quarters of the same quantity.

[6]Here as elsewhere, I will not supply explicit references because I do not wish to appear to be targeting specific persons or works in my criticism. I will be making generic comments about several general areas.

that is why they opt for establishing cognitive, rather than mathematical connections among the "personalities" of rational numbers. The needs of the classroom would seem to be in conflict with the mathematics. At this point, engineering enters.

It turns out that, by changing the mathematical landscape entirely and leaving quotient fields and ordered pairs behind, it is possible to teach fractions as mathematics in elementary school, by finding an alternate mathematical route around these abstractions that would be suitable for consumption by children in grades 5-7. Without going into details, suffice it to say that at least the mathematical difficulties can be overcome, for example, by identifying fractions with certain points on the number line (for this systematic development, see, e.g., Jensen [11], or Wu [25]). What is of interest in this context is that this approach to fractions is specific to the needs of elementary school and is not likely to be taught, ever, in any other situation. In addition, the working out of the basic properties of fractions from this viewpoint is not quite straightforward, and it definitely requires the expertise of a research mathematician. As to the further pedagogical implementation to render such an approach usable in grades 5–7, the input of teachers and educators would be absolutely indispensable.[7] We therefore get to witness how mathematicians and educators are both needed to turn a piece of abstract mathematics into usable lessons in the school classroom. This is customization of abstract theory for a specific human need, and this is engineering at work.

Through this one example of fractions, we get a glimpse of how the principles of mathematical engineering govern the design of a curriculum. Less obvious but of equal importance is the fact that even mathematics education research cannot be disconnected from the same principles. If, for example, a strong mathematical presence had been integral to the research on fractions and rational numbers, it would be very surprising that the research direction would have developed in the direction it did. Compare the quote by Lagemann at the beginning of this article as well as Lagemann [14].

An entirely analogous discussion of customization can be given to any aspect of mathematics education, but we single out the following for further illustrations:

(a) The design of an "Intervention Program" for at-risk students. Up to this point, the methods devised to help these students are largely a matter of teaching a watered-down version of each topic at reduced pace; this is poor engineering from both the theoretical and the practical point of view. In Milgram-Wu [18], a radically different mathematical engineering design is proposed to deal with this problem.

(b) The teaching of beginning algebra in middle school. The way symbols are usually handled in such courses, which necessitates prolix discussions in the research literature of the subtlety of the equal sign, and the way variable is introduced as the central concept in school algebra are clear indications that the algebra we teach students at present has not yet been properly customized

[7] Some teachers who have worked with me are trying out this approach with their students in San Francisco.

for the needs of school students. See the Preface and Sections 1 and 2 of Wu [30], and also Wu [31], for a more detailed account of both the problems and their proposed solutions.

(c) The writing of mathematics standards at the national or state level. This is an example of what might be called "practical optimization problems", which customize the mathematics to meet diverse, and at times conflicting, needs of different clientele. Cf. Klein [13].

The concept of mathematics education as mathematical engineering also sheds some light on Lee Shulman's concept of pedagogical content knowledge ([20]). There has been a good deal of interest in precisely describing the kind of knowledge a teacher should possess in order to be effective in teaching. In the field of mathematics, at least, this goal has proven to be elusive thus far (but cf. Hill-Rowan-Ball [9]), but Shulman's intuitive and appealing formulation of this concept crystallizes the diverse ideas concerning an essential component of good teaching. From the point of view of mathematical engineering, one of the primary responsibilities of a teacher is to customize her mathematical knowledge in accordance with the needs of each situation for students' consumption. This particular engineering knowledge is the essence of pedagogical content knowledge. Although this approach to pedagogical content knowledge does not add anything new to its conception, it does provide a framework to understand this knowledge within mathematics, one that is different from what one normally encounters in educational discussions. It makes explicit at least three components to effective teaching: a solid mathematical knowledge, a clear perception of the setting defined by students' knowledge, and the flexibility of mind to customize this mathematical knowledge for use in this particular setting without sacrificing mathematical integrity.

The idea of customizing mathematics "without sacrificing mathematical integrity" is central to mathematical engineering. In engineering, it is obvious that, in trying to customize scientific principles to meet the needs of humanity, we cannot contradict nature regardless of how great the human needs may be. In other words, one respects the integrity of science and does not attempt anything so foolish as the construction of anti-gravity or perpetual-motion machines. Likewise, as mathematical engineering, mathematics education accepts the centrality of mathematics as a given. Again using the example of teaching fractions, a mathematics educator would know that no matter how one tries to teach fractions, it must be done in a way that respects the abstract meaning of a fraction even if the latter is never used explicitly. If, for instance, an educator catches himself saying that children must adopt new rules for fractions that often conflict with well-established ideas about whole numbers, then he knows he is teaching fractions the wrong way because, no matter what efforts one puts into making fractions intuitive to children, one cannot do violence to the immutable fact that the rational numbers contain the integers as a sub-ring. The need to teach the arithmetic of fractions as a natural extension of the arithmetic of whole numbers has gone unnoticed for far too long, with the result that too many of our students begin to harbor the notion that, after the whole numbers, the arithmetic of fractions is a new beginning. Such bad mathematical engineering in curricular designs is unfortunately a common occurrence.

The only way to minimize such engineering errors is to have both mathematicians and educators closely oversee each curricular design. In fact, if we believe in the concept of mathematics education as mathematical engineering, then the two communities must work together in all phases of mathematics education: Any education project in mathematics must begin with a sound conception of the mathematics involved, and there has to be a clear understanding of what the educational goal is before one can talk about customization. In this process, there is little that is purely mathematical or purely educational; almost every step is a mixture of both. Mathematics and education are completely intertwined in mathematical engineering. Mathematicians cannot contribute to K-12 mathematics education if they are treated as outsiders.[8] They have to work alongside the educators on equal footing in the planning, implementation, and evaluation of each project. But this is far from the reality at present.

For at least three decades now, the mathematics and K-12 education communities in the U.S. have not been on speaking terms in the figurative sense. (Cf. Washington Post [21].) The harm this communication gap has brought to K-12 mathematics education can be partially itemized, but before doing that, let me point out three general consequences of a philosophical nature. The first one is that the isolation of the education community from mathematicians causes educational discussions to over-focus on the purely education aspect of mathematics education while seemingly always leaving the mathematics untouched. The result is the emergence of a subtle mathematics avoidance syndrome in the education community, and this syndrome will be seen to weave in and out of the following discussion of the specific harmful effects of this communication gap. Given the central position of mathematics in mathematical engineering, it would be noncontroversial to say that this syndrome should vanish from all discussions in mathematics education as soon as possible.

The fact that many mathematicians teach mathematics and design mathematics courses throughout their careers seems to escape the attention of many educators. Here is a huge reservoir of knowledge and experience in mathematical engineering on tap. The chasm between the two communities in effect denies educators access to this human resource at a time when educators need all the engineering help they can get.

The final consequence can best be understood in terms of the Darwinian dictum that when a system is isolated and allowed to evolve of its own accord, it will inevitably mutate and deviate from the norm. Thus when school mathematics education is isolated from mathematicians, so is school mathematics itself, and, sure enough, the latter evolves into something that in large part no longer bears any resemblance to mathematics. Correct definitions are not given, or if given, they are not put to use (Milgram-Wu [18], Wu [23], [27] and [29]). The organic coherence of mathematics is

[8]This only tells half the story about mathematicians. See the comments near the end of this article.

no longer to be found (Wu [23]), or when "mathematical connections" are intentionally emphasized, such "connections" tend to be the trivial and obvious kind. Logical deduction becomes an afterthought; proofs, once relegated to the secondary school geometry course, were increasingly diluted until by now almost no proofs at all are found there, or anywhere else in the schools (Wu [26]). And so on. This development naturally brings down the quality of many aspects of mathematics education.

The absence of dialog between the two communities has led to many engineering errors in mathematics education, one of them being the unwelcome presence of mathematically incorrect test items in state and other standardized tests (Milgram [16] and [17]). The same kind of defective items also mar many teachers' credentialing tests (Askey [1] and [2]). A more subtle effect of the absence of mathematical input on assessment is the way test scores are routinely misinterpreted. The low test scores have been used to highlight students' dismal mathematical performance, but little or no thought is given to the possibility that they highlight not necessarily students' achievement (or lack thereof) but the pervasive damage done by defective curricular materials, or even the chronic lack of effective teaching. Such a possibility may not be obvious to anyone outside of mathematics, but to a mathematician, it does not take any research to confirm the fact that when students are taught incorrect mathematics, they learn incorrect mathematics. Garbage in, garbage out. If the incorrect mathematics subsequently shows up in students' test scores, how can we separate the errors due to the incorrect information students were given, from the errors due to students' own misconceptions? A more detailed examination of this idea in the narrow area of school algebra is given in Wu [31]. The need for mathematicians' participation in all phases of assessment is all too apparent.

The lack of collaboration between mathematicians and mathematics educators affects professional development as well. The issue of teacher quality is now openly acknowledged and serious discussions of the problem are beginning to be accepted in mathematics education (cf. Ma [15], and Conference Board of the Mathematical Sciences [6][9]). As a result of the inadequate mathematics instruction teachers receive in K-12, their knowledge of mathematics is, by and large, the product of the mathematics courses they take in college.[10] In very crude terms, the number of such required mathematics courses is too low, and in addition, these courses are taught either by mathematicians who are not in close consultation with teachers, and are unaware as to what is needed in the school classroom, or by mathematics educators who are not professional mathematicians. The former kind of course tends to be irrelevant to the classroom, and the latter kind tends to be mathematically shallow or incorrect. It is only natural that teachers coming out of such an environment turn out to be mathematically ill-prepared.

[9] Whatever reservations one may have concerning the details of its content, it is the fact that such a volume could be published under the auspices of a major scientific organization that is important.

[10] It may be useful to also take note of what may be called "the second order effect" of university instruction: teachers' knowledge of mathematics is also conditioned by their own K-12 experiences, but these teachers' teachers were themselves products of the mathematics courses they took in the university.

Similar woes persist in in-service professional development, thereby ensuring that teachers have little access to the mathematical knowledge they need for their profession. For example, the last decade has witnessed the appearance of case books consisting of actual records of lessons given by teachers.[11] The idea is to invite teachers to analyze these lessons, thereby sharpening their pedagogical sensibilities. In too many instances, however, blatant mathematical flaws in the cited cases are overlooked in the editors' commentaries. This raises the specter of bringing up a generation of teachers who are proficient in teaching school students incorrect mathematics. In this instance, it would appear that the need to respect mathematical integrity in mathematical engineering has been all but forgotten.

The most divisive outcome of the noncommunication between the two communities in the U.S. is undoubtedly the conflict engendered by the new (reform) curricula written in the past fifteen years. I take up this discussion last, because it brings us face to face with some subtle issues about mathematicians' participation in K-12 mathematics education. The prelude to the writing of these curricula is the unchecked degeneration in the mathematical integrity of the existing textbooks from major publishers over the period 1970–1990, a fact already alluded to above. This degeneration triggered the reform spearheaded by NCTM (National Council of Teachers of Mathematics [19]). Rightly or wrongly, the new curricula were written under the banner of the NCTM reform, and the manner in which some of the reform texts were imposed on public schools led eventually to the well-known Math Wars (Jackson [10]). The root of the discontent over these texts is the abundance of outright mathematical errors[12], as well as what research mathematicians perceived to be evidence of a lack of understanding of the mathematics. An example of the latter was the promotion of children's invented algorithms at the expense of the standard computation algorithms in the elementary mathematics curriculum. Although the promotion was partly an overreaction to the way the standard algorithms were often inflicted on school children with nary a word of explanation, it also reflected a lack of awareness of the central importance of the mathematical lessons conveyed by the reasoned teaching of these algorithms.

The "subtle issues" mentioned above stem from the fact that the writing of some of the new reform curricula actually had the participation of a few mathematicians. The first thing to note is that the latter are the rare exceptions to the general noncommunication between the mathematics and education communities. The noncommunication is real. At the same time, these exceptions seem to point to an apparent contradiction: How would I reconcile my critical stance toward these reform curricula with the principal recommendation of this article, namely, that mathematicians be equal partners with educators in the mathematics education enterprise? The answer is that there is no contradiction at all. The participation by mathematicians is, in general terms, a prerequisite to any hope of success in K-12 mathematics education, but in no way

[11] Let it be noted explcitly that I am discussing the case books in K-12 mathematics education only.

[12] These errors tend to be different from the earlier ones to be sure, but errors they are.

does it guarantee success. It is helpful in this context to recall similar discussions that routinely took place some eight years ago when some mathematicians first went public with the idea that mathematics teachers must have a solid content knowledge. The usual rejoinder at the time was that "knowing mathematics is not enough (to be a good teacher)". This is a common confusion that mistakes a necessary condition for a sufficient condition.[13] There is no quick fix for something as complex as mathematics education. Getting mathematicians to fully participate is only the beginning; the choice of the mathematicians and the hard work to follow will have a lot to say about the subsequent success or failure.

It is appropriate at this point to recall what was said at the beginning of the article about the power structure of mathematics education: thus far, educators get to make the decisions. Granting this fact, I should amplify a bit on the difficulties of choosing the right mathematicians for education work. Mathematicians have a range of background and experiences and, consequently, often have a range of opinions on matters of education as well. It is important that the range of these opinions be considered in all aspects of education. Many of the less happy incidents of the recent past in K-12 mathematics education were the result of choosing mathematicians of a particular persuasion. In addition, educators must make their own judgement on which among the mathematicians interested in K-12 are knowledgeable about K-12. Among the latter, some possess good judgment and leadership qualities while others don't. Educators must choose at each step. If there are algorithms for making the right choices, I don't happen to know them.

Every mathematician potentially has something to offer in K-12 mathematics education: even an occasional glance at textbooks to check for mathematical correctness can be very valuable. However, if mathematicians want to participate in serious educational work in K-12, what must they bring to the table? I believe the most important thing is the awareness that K-12 mathematics education is not a subset of mathematics, and that there is quite a bit to learn about the process of customization that distinguishes K-12 mathematics education from mathematics. In particular, much (if not most) of the mathematics they teach in the university cannot be brought straight to the school classroom (Wu [22]; Kilpatrick et al. [12], Chapter 10 and especially pp. 375–6), but that it must first go through the engineering process to make it suitable for use in schools. If I may use the example of fractions once again, mathematicians interested in making a contribution to K-12 may find it instructive to get to know the reason that something like "equivalence classes of ordered pairs of integers" is totally opaque to students around the age of twelve. They would also want to know the reason that students of that age nonetheless need a definition of a fraction which is as close to parts-of-a-whole as possible. They should also get to know the appropriate kind of mathematical reasoning for students in this age group, because they will ultimately be called upon to safeguard such reasoning in the curriculum and

[13] And need I point out, there are some who intentionally use this confusion to reject that mathematical content knowledge is important for teachers, or that getting mathematicians to participate in mathematics education is critical for its success.

assessment for these students.

Mathematicians may regard school mathematics as technically primitive (in the sense of skills), but they must take note of its conceptual sophistication (Jensen [11]; Wu [24], [25] and [30]; cf. also Aharoni [1]). Above all, they must know that school mathematics is anything but pedagogically trivial: There is absolutely nothing trivial about putting any material, no matter how simple, into a correct mathematical framework so that it may be profitably consumed by school students. Mathematicians who want to contribute to K-12 mathematics education have to be constantly on the alert to ensure that the minimum requirements of their profession – the orderly and logical progression of ideas, the internal cohesion of the subject, and the clarity and precision in the presentation of concepts, – are still met in mathematics education writings. This is no easy task. If mathematicians want to enter K-12 mathematics education as equal partners with educators, then it is incumbent upon them to uphold their end of the bargain by acquiring this kind of knowledge about mathematical engineering.

The concept of mathematics-education-as-mathematical-engineering does not suggest the creation of any new tools for the solution of the ongoing educational problems. What it does is to provide a usable intellectual framework for mathematics education as a discipline, one that clarifies the relationship between the mathematics and the education components, as well as the role of mathematicians in mathematics education. For example, it would likely lead to a better understanding of why the New Math became the disaster that it did. Most importantly, this concept lays bare the urgent need of the mathematical presence in every aspect of K-12 mathematics education, thereby providing a strong argument against the self-destructive policy of keeping mathematicians as outsiders in mathematics education. The chasm between mathematicians and educators must be bridged if our children are to be better served. I am cautiously optimistic[14] that there are enough people who want to rebuild this bridge (cf. Ball et al. [4]), all the more so because the indications are that the NCTM leadership is also moving in the same direction. I look forward to a future where mathematics education is the joint effort of mathematicians and educators.

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It gives me pleasure to thank them all.

[14] In January of 2006.

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The role of mathematicians in K-12 education: a personal perspective

by *Shiu-Yuen Cheng*

This draft was written after I read Prof. H. Wu's draft on "How mathematicians can contribute to K-12 mathematics education". I therefore have the advantage of adopting the same terms and scope of discussions in writing this draft. For example, I will be using Prof. Wu's definition of the word "mathematician" to mean "research mathematicians". Also, I am impressed and I agree with Prof. H. Wu's philosophical idea of regarding mathematics education as mathematical engineering. I will in the following outline the roles that mathematicians can play for the enhancement of K-12 education. Frequently, I will come back to Prof. Wu's idea of mathematical engineering so that we can do a good job.

The main factors for providing an effective mathematics education are curriculum design, teacher competence and assessment methods. Among these three factors, I think the most important one is teacher competence. I think this is the factor that mathematicians can contribute the most. The processes of designing the curriculum and assessment mechanism vary from place to place and are greatly influenced by the local bureaucratic and political system. In most places, mathematicians do not get to play much of a role in the design of curriculum and the assessment mechanism. However, this does not mean that we should fold our hand and watch on the sideline. We should always engage in these two factors and make our contributions whenever possible. On the other hand, any curriculum design or assessment method in mathematics would need or welcome mathematicians' stamp of approval. Mathematicians will definitely be involved but we have to vigilantly and patiently engage in the process.

In Hong Kong, the relation between educators and mathematicians is much better than in the US. However, the educators do not call the shot. Instead, the government officials set the agenda and play the most influential role. On paper, it does not seem so because things are supposedly done through committees consisting of teachers, principals, educators and mathematicians. The government officials serve as secretariats in the committees. As the committee members are all busy people taking time off from their work to attend the committee meetings, the secretariat then get to draft all the papers and the agenda. They naturally become most influential. Moreover, as a consequence of the composition of the committees mathematicians are minority. To make things worse, they have few allies. The teachers, school principals in the committee usually assume the mathematicians in the committee have a secret agenda to tailor the curriculum for attracting students to be mathematics majors. Additionally, the educators talk the language of education officials. Their inputs are more helpful to the education officials in filling the reports with popular education jargons. It is then natural that mathematicians' views usually do not prevail. Instead, some compromise can usually be reached if mathematicians engage in the process.

I believe that teacher competence is most important factor as teachers are at the frontline implementing the curriculum and delivering the mathematics education. No matter how hard we work, usually the curriculum and assessment mechanism are far from perfect. A competent teacher can exercise discretions to compensate the inconsistencies and incompleteness of the curriculum and make them work. On the other hand, a teacher who has little confidence and competence in subject knowledge can easily turn a well-designed curriculum or assessment mechanism into disasters. In the area of teacher competence, mathematicians can contribute in two main areas: the university curriculum for mathematics teacher program, and courses and workshops for in-service mathematics teachers. Mathematicians can play major roles in these two areas and can get more colleagues to participate. However, we usually do not pay much attention or do not do the right thing. It was pointed out by Prof. H. Wu and many others that the university curriculums for mathematics majors do not serve the purpose of providing the necessary understanding to be a competent mathematics teacher. The main reason is that the curriculum for mathematics majors is designed with an aim to train research mathematicians. As for courses and workshops for in-service teachers, we need more colleagues to participate and contribute. The ball is in our court but so far we have not made the right play.

In Hong Kong, about fifteen to twenty percent of mathematics graduate become mathematics teachers. This is not a small percentage and is in fact higher than the percentage of students going for postgraduate study in mathematics. However, the curriculum for mathematics major does not offer much help for those who will pursue the career of a mathematics teacher. Mainly, most mathematicians do not see the necessity of designing and offering some new courses to provide a profound understanding of school mathematics. It is assumed that our courses in abstract algebra, analysis and geometry will do the job and hence nothing needs to be done. As for courses and workshops for in-service teachers, the sad thing is that Hong Kong government does not provide much of this kind of opportunity. Mathematicians have to shoulder this task on a volunteer basis. There are mathematicians willing to contribute but in order to make it sustainable we need to convince the government and the mathematics community the importance of providing courses and workshops for deepening teachers' understanding of the subject knowledge. To do this effectively, we need to communicate well to the community about the concerns of mathematicians about mathematics education. In many places, people are alarmed because "Johnny can't add". In Hong Kong, Johnny can add! In fact, Johnny can do fractions and decimals quite well. In many international studies about the mathematics attainment of school students, Hong Kong routinely occupies one of the top positions. On paper, we should congratulate ourselves and should not even attempt to touch the system as things are not broken. However, anyone in the university or familiar with the situation of the Hong Kong school system knows that the Hong Kong mathematics education is far from achieving the goals. We are able to train our students to do arithmetic and some simple algebra at a tremendous cost. In the process, we suffocate students' creativity and motivation for learning. So far we have not been successful in documenting and communicating our concerns to the Hong Kong public and the government. Mathematics education is then getting little resource from the government as it is doing quite well comparing to our language education. I believe communicating effectively to the public about our views for enhancing mathematics education is crucial and should fall into one of the sub-areas of Prof. Wu's framework of Mathematics Engineering. The banner of Mathematics Engineering is useful for setting a clear goal and rallying support of our fellow mathematicians to contribute to mathematics education.

The role of mathematicians in K-12 mathematics education

by *U. Kirchgraber and K. Osterwalder*

We begin with a few remarks on the Swiss educational system and on teaching and learning of mathematics in Switzerland. Then we focus on teacher training in Mathematics at the Swiss Federal Institute of Technology (ETH). Finally we sketch an answer to the question posed to the panel.

In international comparative studies like TIMS and PISA Swiss students have demonstrated reasonable achievements in Mathematics. Without overestimating such results[15] one may wonder whether some specific features of the Swiss educational system might be responsible for this relative success and which measures could serve a further improvement of the results. As we will see some of the possible explanations are related to the topic of the panel.

It is well known Switzerland is not rich in natural resources. This is usually claimed to be one of the major reasons why education is quite highly valued in this country, with a number of important implications: the vast majority of schools are public, teachers on all levels are well trained, the profession of teacher is quite respected, teachers (on all levels) are well paid, schools are well equipped, school buildings are kept in good shape.

The Swiss educational system leaves considerable freedom to teachers on all levels and in particular in upper secondary school, to which we will refer to as Gymnasium[16] level[17]. Gymnasium teachers in general and Gymnasium Mathematics teachers in particular have to follow a certain core curriculum. Yet beyond this guide line they are fairly free to include additional topics, there are hardly any restrictions concerning the type of pedagogy adopted, and teachers are even quite free as to the number and type of tests and examinations they will administer. As to the final examination, in many schools every single mathematics teacher is free to assign a selection of, say, 4-8 problems of his or her choice and depending on the topics he or she has covered in class to his or her students on which they will work during 4 hours. Eventually he or she will correct and grade these works.

ETH offers Gymnasium Teacher Training Programs in the following fields: Biology, Chemistry, Earth Sciences, Mathematics, Physics and Sports. In the following we discuss some features of the Gymnasium Mathematics Teacher Training Program (GMTTP).

A prerequisite for completing the GMTTP is a Master's Degree in Mathematics, though the students are permitted to start with the GMTTP in the third year of the Bachelor's program. Average duration time for completing the GMTTP is six months[18], if studied full time.

The fact that Swiss Gymnasium Mathematics teachers must hold a Master's Degree in Mathematics has – we suppose – far reaching professional and psychological consequences. Having completed a Master's program has at least two implications which, we think, are important for a future Mathematics teacher: a) During the first two years of studies Mathematics students encounter many topics they have seen before yet dealt with in way that is qualitatively very different from what they had

[15] Compared to the host of highly sophisticated tools to measure many quantities in the Sciences and in particular in Physics that have evolved since the time of Gallilei, measuring effectiveness of teaching and learning and similar variables is probably still in its infancy, yet is a fascinating and challenging enterprise.

[16] There are some 150 Gymnasia in Switzerland, every year some 15000 students graduate from the Gymnasia, between 1800 and 2300 enroll at ETH.

[17] This corresponds to grades 9–12.

[18] Starting in fall 2006 federal requirements request studies twice as long. ETH's GMTTP will be extended accordingly and will be renamed as "Master of Advanced Studies in Secondary and Higher Education in Mathematics".

experienced previously. b) In the third year of the Bachelor's and during the Master's program they are exposed to advanced fields, an indispensable experience for gaining a faithful picture of what Mathematics is about.

Based on four and a half years of studies these teacher students can at least potentially be expected to dispose of a degree of mathematical expertise and mathematical maturity which is covered by the terms "content knowledge" and/or "deep understanding" in the Mathematics Education research literature.

As to some practical implications: Being trained as full fledged mathematician a Gymnasium Mathematics teacher may leave school after a few years and start a career in Industry or elsewhere, or vice versa. Therefore requiring a Master's Degree as a prerequisite for teacher training has the benefit of not excluding Gymnasium Mathematics teachers in an early stage of their professional development from the full scale of professional opportunities offered to mathematicians nowadays.

The GMTTP includes courses in the Educational sciences, in Mathematics Education (Didactics of Mathematics), (a small amount of) guided teaching practice, and a fourth component, called Specialized Mathematics courses with an Educational Focus. It is of utmost importance that these components are excellently tuned and multiply intertwined. Moreover they should be accompanied by plenty of student activities[19].

Since the late eighties the Educational course at ETH was designed and continuously updated. The basic concept was to make available both to the Science and Mathematics Educators as well as to their teacher students research grounded results from areas such as psychology, the cognitive sciences, etc. Over the years quite a number of teaching techniques and teaching methods were implemented and probed since they are known – on the basis of meta analyses[20] – to enhance learning[21]. Guided Learning programs[22] for instance are self-contained study materials for pupils covering a learning unit of some 3–30 lessons with the following features: Precisely defined prerequisites, well structured and comprehensibly written explanations, explicitly stated learning goals, adjunct questions and their answers, learning aids, chapter tests to fulfill the so-called Mastery Learning Principle. According to Kulik, Kulik and Bangert-Drowns[23] the effect size of this type of teach ware is of the order of 0.5 in Mathematics and of the order of 0.6 in the Sciences. A few examples of Guided Learning Programs in Mathematics and the Sciences (in German) can be found on www.educeth.ethz.ch[24].

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[19] As research has shown, just attending lectures has little impact on future teaching.

[20] See for instance Fraser B. J., Walberg H. J., Welch W. W., Hattie J. A., Syntheses of Educational Productivity Research. *International J. of Educational Research* **11** (1987), 145–252; Walberg, H. J., Productive Teaching and Instruction: Assessing the Knowledge Base. University of Illinois at Chicago, School of Education, 1988, 18 p, mimeographed.

[21] I.e. they have noteworthy effect sizes.

[22] In German: Leitprogramme.

[23] Kulik F. S., Kulik J. A., Bangert-Drowns R. L., Effectiveness of Mastery Learning Programs: A Meta-Analysis. *Review of Educational Research* **60** (1990), 265–299.

[24] EducETH, a service of ETH to the Public, is ETH's educational server providing teaching materials primarily for upper secondary schools.

In the Mathematics Education courses the thrust is on domain specific aspects of the teaching and learning enterprise. Of course, goals, standards, competencies to be achieved are discussed, subject matter analysis with diverse approaches to selected topics is of central concern and textbooks[25] are analyzed. The teacher students are exposed to Mathematics Education research concepts that prove useful to explain certain phenomena, for instance Talls's and Vinner's distinction between concept definition and concept image which helps to understand students' misconceptions of the notion of function. Topics like the "Expert Blind Spot"-Hypothesis[26], or the influence of teacher's pedagogical content beliefs on learning outcomes[27], and many others are treated. Videos and their transcriptions are analyzed to provide insight into the unpredictability and fragility of learning processes, among other things.

We now turn to the Specialized Mathematics courses with an Educational Focus already mentioned before. It is by now generally accepted that transfer achievements quite often do not emerge automatically. F. Weinert, summarizing years of research at the Max Planck Institut für psychologische Forschung in Munich, explains it roughly speaking as follows. Knowledge a learner acquires systematically – for instance in Mathematics courses as they are usually organized – is likely to be structured and organized in the learners brain in a way not easily retrievable, amenable if the learner is put in a problem situation in which he/she should apply this body of knowledge. Thus, knowledge, which is available in principle, remains inert and unused, though it would be useful and even necessary to handle a certain situation. Weinert's conclusion: To build up an intelligent, flexibly applicable knowledge base the learner needs both, systematic as well as situated learning.

The Specialized Mathematics courses with an Educational Focus take place in the third year (in Switzerland: the last year) of the Bachelor's and during the Master's program. They are open to all students in the Bsc/Msc Mathematics program, but are compulsory for candidates in the GMTTP. These courses with an Educational focus were installed many years ago at ETH (long before there were courses on Mathematics Education!) and have their origin in lectures given by Felix Klein early in the 20th century in Göttingen under the title "Elementarmathematik vom höheren Standpunkt" (elementary mathematics from an advanced point of view) and directed to future Gymnasium Mathematics teacher.

The Specialized Mathematics courses with an Educational Focus serve several goals. Very much in the spirit of Klein's concept they attempt to narrow the gap between Gymnasium Mathematics and University Mathematics. Take a core topic in Mathematics, present at all levels: equations. Linear equations are a topic in grade 9,

[25]After what has been said earlier in this paper it will not come as a big surprise for the reader that Swiss Gymnasium Mathematics teacher are not obliged to use any particular text books. Many in fact do not use a textbook at all but use a variety of sources to assemble handouts, etc., for their students.

[26]See: M. J. Nathan, A. Petrosino: Expert Blind Spot among Pre-service Teachers. *Amer. Educ. Res. J.* **40** (2003), 905–928.

[27]See: Staub, F. C. and Stern, E., The Nature of teachers' Pedagogical Content Beliefs Matters for Students' Achievement Gains. *J. Educational Psychology* **93** (2002), 344–355.

in a course on Linear Algebra in the first year of the Bachelor's program, as well as in specialized courses on Numerical Linear Algebra: Relate the various aspects and draw conclusions for the future teaching of linear equations in grade 9[28]!

Pupils usually encounter nonlinear equations first in connection with the quadratic equation. Most emphasis is usually put on reducing a general quadratic equation[29] to a "purely quadratic" equation[30]. It is of course a marvelous discovery that arbitrary quadratic equations can be reduced to purely quadratic ones. It is a challenging design task for teacher students to compose a series of assignments that guides pupils to discover this phenomenon by themselves.

Yet from a more general point of view the question of solving purely quadratic equations is even more intriguing. One encounters a pattern that is prevalent in (University) Mathematics: Equations are not always solvable. More often than not mathematicians have to invent a setting such that the equation becomes solvable: Loosely speaking – mathematicians make equations solvable! A second such instance comes up when complex numbers are invented to make all quadratic equations solvable with the totally unexpected benefit that in this setting all polynomial equations have solutions.

The Fundamental Theorem of Algebra brings up another very interesting phenomenon: we may be able to prove that an equation has a solution and even that this solution is unique without being able to compute the solution. If complex numbers are treated at the Gymnasium[31] level an intuitive proof of the Fundamental Theorem of Algebra can be offered to Gymnasium students. In a Specialized Mathematics course with an Educational Focus dedicated to equations, teacher students would not only design a learning unit for pupils around such a heuristic proof, but learn in addition how such a proof is made rigorous (not an easy task!), topics like Rouché's Theorem would have to be discussed and an introduction to the Brouwer and Leray–Schauder Degree theory with applications to periodic solutions of differential equations would allow for a glance of the breadth of the field.

Another aspect to which Specialized Mathematics courses with an Educational Focus can contribute concerns curricular development. School Mathematics curricula are often blamed for covering material only that was invented centuries ago. Of course, most subjects that are hot research topics in Mathematics are far too remote and far too specialized for being accessible at the Gymnasium level. Yet there are marvelous exceptions: The Diffie-Hellman Key Exchange and RSA Cryptography, invented in

[28]Maybe you conclude that the 9th grade program on linear equation should contain a modest introduction to Computerized Tomography, as we did, see the Leitprogramm entitled "Gleichungen" at www.educeth.ethz.ch. Maybe you conclude that the program, in addition to Gaussian elimination, should include a homeopathic introduction to solving linear equations by iteration (a topic you might touch on again, when you treat Banach fixed point iteration in one dimension in connection with Kepler's equation). Maybe you conclude that the question of what it means that two linear systems of equations are equivalent, and how one obtains equivalent systems from a given one, deserves to become (a small?) Mathematics Education research project.

[29]I.e. one including a linear term with respect to the unknown.

[30]I.e. one in which the linear term is absent.

[31]Gymnasia have various different profiles in Switzerland. Some concentrate on Mathematics and Physics. There complex numbers are treated.

the late seventies, are well suited to give 10th graders an idea of the mathematical enterprise[32].

Euler buckling is probably the earliest example of a bifurcation problem. It was only during the last two or three decades, however, that bifurcation theory became a systematically developed branch of analysis. In a Specialized Mathematics course with an Educational Focus dedicated to an introduction to bifurcation theory as background it is well possible that teacher students adapt some of the material for Gymnasium students, hereby heavily drawing on a classic school subject: the study of the geometrical properties and graphing of functions defined by simple expressions.

Finally we mention ill-posed inverse problems. This again is a relatively new field of Applied Mathematics. It is of great theoretical and practical interest, the way ill-posed inverse problems are treated mathematically is surprising and they lend themselves outstandingly for treatments on various different levels[33].

We expect that Specialized Mathematics courses with an Educational Focus deepen the teacher students' mathematical expertise, that they strengthen the link between University and Gymnasium level Mathematics, that they contribute to develop the secondary school Mathematics curricula, that they support the prospective teachers to teach Mathematics at the same time more mathematically and in such a way that their students can learn to value Mathematics as a human activity and for its significance in our world.

What is eventually the role of mathematicians in K-12 in our system?

Research mathematician can and do contribute in a number of ways. Via the Bachelor's and Master's program they shape lastingly the knowledge base, the picture and the skills our teacher students develop. They can substantially contribute to the Specialized Mathematics courses with an Educational Focus. They can contribute to the design of substantial teaching units.

In Mathematics Education research highly interesting developments have just begun. We mentioned the paper by Staub and Stern entitled: "The Nature of teachers' Pedagogical Content Beliefs Matters for Students' Achievement Gains." Another paper in the same realm is by Hill, Rowan and Ball[34]. It is entitled: "Effects of teachers' mathematical knowledge for teaching on student achievement." These papers provide results on primary school teachers and primary school pupils. We certainly need many more results on "what has which impact on student learning" and very much so in higher grades. In fact, very little seems to be known as to the upper Gymnasium level.

[32] Here are some aspects: Fermat's (little) Theorem on which RSA cryptography is based, though elementary, is far from being obvious. The way it is proven illustrates the power of mathematical ideas. 350 years after its discovery it became the key ingredient to affirmatively answer a questions one hardly dares to ask: Is it possible that two persons, who cannot communicate but publicly, can agree on keys which permit them to exchange messages which cannot be decoded except by the person who is entitled to read the message?

[33] See Kirchgraber, U., Kirsch, A., Stoffer, D. : Schlecht gestellte Probleme – oder wenn das Ungenaue genauer ist. *Math. Semesterber.* **51** (2004), 175–2005.

[34] Hill, H. C., Rowan, B., Ball, D. L.: Effects of teachers' mathematical knowledge for teaching on student achievement. *Amer. Educ. Res. J.* **42** (2005), 371–406.

Earlier we noted that educational measurement techniques are probably still in their infancy. How can we suitably measure mathematical achievements, teachers' pedagogical content knowledge, the nature of teachers' pedagogical content beliefs, and many more variables of this type? This is certainly a field, where research mathematician can and should contribute.

Research mathematicians in Switzerland are welcome as members in school boards, and/or as experts in the final examinations at Gymnasium Schools. Research mathematicians are welcome to offer lectures to in-service teachers or to participate in study weeks for Gymnasium students, or to visit schools and give talks.

To summarize: The main contribution of research mathematicians to the second half of K-12 is to train Mathematics teachers as knowledgeable mathematicians and to develop with them methods to narrow the gap between "Gymnasium Mathematics" and University Mathematics. Other possible contributions are manifold, crucial and indispensable.

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