Axioms for a mixed Weil cohomology theory

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In this short communication, I will present a joint work with Denis-Charles Cisinski (cf [1]) on cohomology theories in algebraic geometry via the motivic homotopy theory of V. Voevodsky and F. Morel (see [2] for the starting definition).

Let us fix a base field k and a coefficient field K. Consider $\mathcal{V}$ the category of smooth affine algebraic k-schemes. I will focus on the situation of a cohomology theory H^* defined on $\mathcal{V}$ with values in $\mathbb{N}$-graded K-algebras. Our main result is that if H^* can be defined via a presheaf of differential graded K-algebra on $\mathcal{V}$, then <u>the following properties</u> :

The functor H^* is additive, $H^*(\mathbb{A}^1_k)$ is concentrated in degree 0 equal to K, $H^*(\mathbb{G}_m)$ is concentrated in degree 0 and 1 equal to $K \oplus K$, H^* satisfies the Brown-Gersten property (i.e. descent for the Nisnevich topology), H^* satisfies the Künneth property,
<u>implies the followings</u> :

The cohomology H^* admits a canonical extension to smooth k-schemes which is an oriented cohomology theory in the sense of [3]. For any smooth k-scheme X, $H^*(X)$ is a finite dimensional K-vector space, H^* satisfies Poincaré duality for smooth projective schemes. There exists a canonical cohomology with compact support H^*_c attached to H^* and for any smooth k-scheme X of dimension d, $H^*(X)$ is the dual of $H^{2d-*}_c(X)$.

The proof really consists of representing the cohomology H^* by an object E in a suitable triangulated category of motives $\mathcal{D}$ (a variant of the stable homotopy category of schemes) : we have for any $X \in \mathcal{V}$, $H^n(X) = \mathrm{Hom}_{\mathcal{D}}(M(X), E[n])$. The technical point in the proof is to show the functor $\mathrm{R}\mathsf{Hom}_{\mathcal{D}}(K, E \otimes .)$ is monoidal where $\mathrm{R}\mathsf{Hom}_{\mathcal{D}}$ stands for the differential graded homomorphisms in $\mathcal{D}$.

This theorem applies particularly to algebraic De Rham cohomology in characteristic 0 and rigid cohomology in characteristic p.

If times allow it, we will also present a strategy to show the previous hypothesis implies proper descent for the extended cohomology H^*. If the strategy works, then we can associate to H^* a realisation functor of the triangulated category of motives defined by Voevodsky.

[1] Cisinski D.-C. and F. Déglise, Künneth formula, duality and finiteness, *Preprint* march 2006, http:// www.math.univ-paris13.fr/ deglise/preprint.html

[2] Morel, F. and Voevodsky, V., A^1-homotopy theory of schemes. In *Inst. Hautes Études Sci. Publ. Math.* **90**, (1999), 45–143.

[3] Levine, M. and Morel, F. Cobordisme algébrique. I. In *C. R. Acad. Sci. Paris Sér. I Math.* **332**, (2001), no. 8, 723–728.